Robust imputation procedures in the presence of influential units in surveys

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Influential units

- In practice, we often face the problem of influential values in the selected sample
- An influential unit is a legitimate unit of the finite population. It is not a measurement error:
  - Gross error;
  - Measurement errors are detected at the editing stage and are treated either manually or by some form of imputation.
- Assumption: Influential units are legitimate observations (not errors)
- Survey statistics are typically sensitive to the presence of influential units

Influential units

- Including or excluding an influential unit in the calculation of survey statistics can have a dramatic impact on their magnitude

  $\longrightarrow$ Their presence in the sample tends to make classical estimators very unstable

  $\longrightarrow$ large variance

- Common issue in business surveys that collect economic variables whose distributions are highly skewed
  - Influential units are often associated with very large values or very large errors
  - Stratum jumpers: may combine a very large value and a large sampling weight

Influential units

- In the presence of influential units, an imputed estimator of a population total:
  - is (approximately) unbiased provided that the imputation model is correctly specified
  - may have a very large variance
- Treatment of influential values: produces stable but biased estimators $\longrightarrow$ trade-off between bias and variance
- Objective: reduce the influence of units that have a large influence
- Our hope: the mean square error of the robust version is smaller than that of the corresponding classical estimator
- How to impute/estimate in the presence of influential units?

The setup

- U: finite population of size N;
- Goal: estimate a population total of a survey variable y:

$$t_y = \sum_{i \in U} y_i$$

- S: sample of size n selected according to a given sampling design $p(S)$;
- I_i: sample selection indicator such that $I_i = 1$ if $i \in S$, and $I_i = 0$, otherwise;
- Design-unbiased (or p-unbiased) estimator of t_y:

$$\widehat{t}_{HT} = \sum_{i \in S} d_i y_i$$

  - $d_i = 1/\pi_i$: design weight attached to unit i;
  - π_i: first-order inclusion probability attached to unit i

The setup

- The survey variable Y is prone to missing values.
- Let r_i be the response indicator such that

$$r_i = \begin{cases} 1, & \text{if } y_i \text{ is observed,} \\ 0, & \text{if } y_i \text{ is missing.} \end{cases}$$

- Set of respondents: $S_r = \{i \in S; r_i = 1\}$.
- Set of nonrespondents: $S_m = \{i \in S; r_i = 0\}$.
- Imputed estimator of t_y:

$$\widehat{t}_I = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i y_i^*,$$

where y_i^* is the imputed value for the missing y_i.

Deterministic linear regression imputation

- x: vector of fully observed variables
- Imputation model

$$y_i = \mathrm{x}_i^\top \boldsymbol{\beta} + \epsilon_i,$$

such that

$$\mathbb{E}(\epsilon_i \mid \mathrm{x}_i) = 0, \mathbb{E}(\epsilon_i \epsilon_j \mid \mathrm{x}_i, \mathrm{x}_j) = 0, i \neq j \text{ and } \mathbb{V}(\epsilon_i \mid \mathrm{x}_i) = \sigma^2 \phi_i$$

with $\phi_i > 0$ (known)

- Estimator of $\boldsymbol{\beta}$ based on the responding units:

$$\widehat{\mathrm{B}}_{\mathrm{WLS}} = \left(\sum_{i \in S_r} d_i \mathrm{x}_i \phi_i^{-1} \mathrm{x}_i^\top \right)^{-1} \sum_{i \in S_r} d_i \mathrm{x}_i \phi_i^{-1} y_i$$

- Imputed value: $y_i^* = \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{WLS}}$

Imputed estimator

- Estimator of t_y after deterministic linear regression imputation:

$$\widehat{t}_{I,WLS} = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{WLS}}$$

- If the first moment of the imputation model is correctly specified, we have

$$\mathbb{E}_m \mathbb{E}_p \mathbb{E}_q (\widehat{t}_{I,WLS} - t_y) = 0.$$

- That is, the estimator $\widehat{t}_{I,WLS}$ is *mpq*-unbiased for t_y.
- However, $\widehat{t}_{I,WLS}$ may be inefficient in the presence of influential units.

Two methods commonly used in practice

- Robust regression: Replace the estimator $\widehat{\mathrm{B}}_{\mathrm{WLS}}$ by a robust version $\widehat{\mathrm{B}}_{\mathrm{R}}(c)$; for instance an *M*-estimator based on the Huber function;
  $\longrightarrow$ $\widehat{\mathrm{B}}_{\mathrm{R}}(c)$ is solution of

$$\sum_{i \in S_r} \psi_c \left(\frac{y_i - \mathrm{x}_i^\top \boldsymbol{\beta}}{\sqrt{\phi_i}\widehat{\sigma}} \right) \frac{\mathrm{x}_i}{\sqrt{\phi_i}} = 0,$$

  where $\psi_c(\cdot)$ is the so-called Huber function and c is a tuning constant.
- Typically, the value is set to 1.345 (as in classical statistics)
- Imputed value: $y_i^* = \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(1.345)$
- Other ψ-functions: Biweight, Andrew, etc.
- Other estimators: GM, MM, LTS estimators, etc.
- Objective of robust regression : describe the behavior of the inliers (the non-outliers)

Huber function

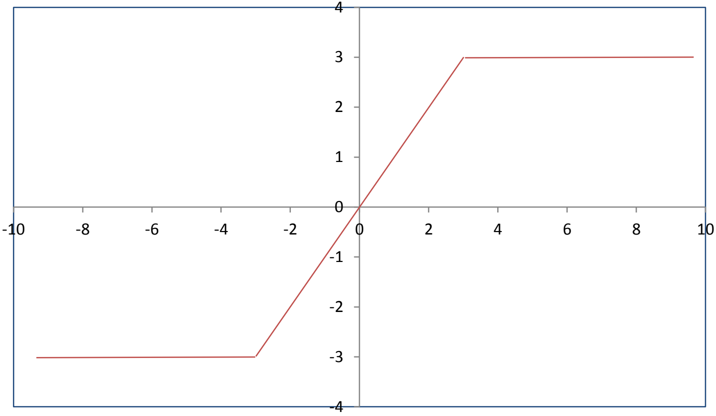


Figure 1: Huber function with $c = 3$

Two methods commonly used in practice

- Excluding outliers: Identify the influential units (usually by an outlier detection method), remove these units and obtain a predicted value obtained by fitting the customary linear regression model
- Imputed value: $y_i^* = \mathrm{x}_i^\top \widehat{\mathrm{B}}^*_{\mathrm{WLS}}$, where

$$\widehat{\mathrm{B}}^*_{\mathrm{WLS}} = \left(\sum_{i \in S_r} \omega_i \mathrm{x}_i \phi_i^{-1} \mathrm{x}_i^\top \right)^{-1} \sum_{i \in S_r} \omega_i \mathrm{x}_i \phi_i^{-1} y_i,$$

  where $\omega_i = d_i$ if i is not discarded and $\omega_i = 0$ if i is discarded.
- Underlying assumption: the discarded respondent y-values are unique; i.e., they do not represent similar non-respondents $\longrightarrow$ nonrepresentative respondents

A simulation study

Are these methods satisfactory?

- We repeated 10, 000 iterations of the following process:
  (1) A population U of size $N = 10,000$ was generated, with one survey variable Y and one covariate X using a mixture of normal distribution with a proportion of outliers equal to 5%;
  (2) A sample S of size $n = 100; 200; 500$ was selected from U according to simple random sampling without replacement;
  (3) Nonresponse to Y was generated according to a uniform nonresponse mechanism with $p_i = 50\%$ for all i;
  (4) Missing values were imputed using 3 imputation procedures.

A simulation study: Point estimators

We computed three types of imputed estimators:

- Non-robust estimator:

$$\widehat{t}_{I,WLS} = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{WLS}}$$

- Based on robust regression:

$$\widehat{t}_I(c) = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(c)$$

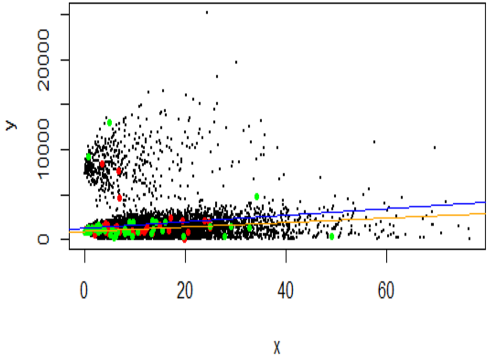
  We used the Huber function with $c = 0.1; 1.345; 2.5$.

- Excluding the outliers:

$$\widehat{t}^*_{I,WLS} = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}^*_{\mathrm{WLS}}$$

  We used the Cook distance with threshold $c = 4/(n-3)$ and studentized residuals with $c = 2; 2.5; 3$.

A simulation study: Asymmetric outliers



$n = 100$

- Respondent
- Nonrespondent
- Nonsampled unit

------- Least squares regression
------- Robust regression

A simulation study: Results

- Monte carlo percent relative bias :

$$\mathrm{RB}(\widehat{t}_I) = \frac{\mathbb{E}_{MC}\left(\widehat{t}_I - t_y\right)}{t_y} \times 100$$

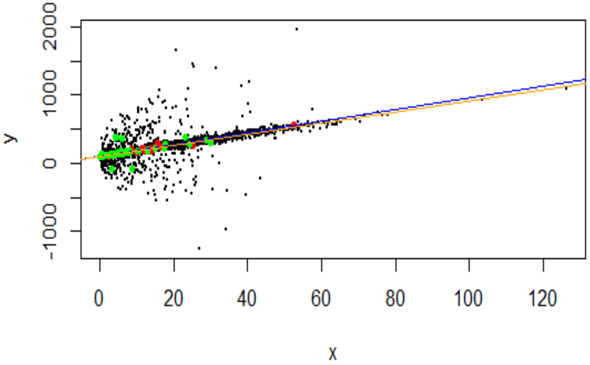
- Relative efficiency:

$$\mathrm{RE} = 100 \times \frac{\mathrm{MSE}_{MC}(\widehat{t}_I)}{\mathrm{MSE}_{MC}(\widehat{t}_{I,WLS})}$$

	WLS	Robust regression			WLS (Exclude outliers)			
n		$c = 0.1$	$c = 1.345$	$c = 2.5$	Studentized $c = 2$	Studentized $c = 2.5$	Studentized $c = 3$	Cook distance
100	-0.0 (100)	-11.5 (78)	-10.7 (73)	-9.7 (70)	-9.3 (82)	-8.3 (84)	-7.5 (86)	-7.5 (87)
200	-0.2 (100)	-11.6 (128)	-10.8 (116)	-9.5 (102)	-9.1 (113)	-7.9 (111)	-6.9 (109)	-7.1 (110)
500	-0.2 (100)	-11.6 (260)	-10.8 (230)	-9.4 (190)	-8.5 (189)	-7.1 (166)	-6.0 (149)	-6.2 (156)

Table 1: Monte Carlo percent relative bias and Monte Carlo relative efficiency of several estimators

A simulation study: Symmetric outliers



n =100

- Respondent
- Nonrespondent
- Nonsampled unit

------- Least squares regression

------- Robust regression

A simulation study

- Monte carlo percent relative bias :

$$\mathrm{RB}(\widehat{t}_I) = \frac{\mathbb{E}_{MC}\left(\widehat{t}_I - t_y\right)}{t_y} \times 100$$

- Relative efficiency:

$$\mathrm{RE} = 100 \times \frac{\mathrm{MSE}_{MC}(\widehat{t}_I)}{\mathrm{MSE}_{MC}(\widehat{t}_{I,WLS})}$$

	WLS	Robust regression			WLS (Exclude outliers)			
n		$c = 0.1$	$c = 1.345$	$c = 2.5$	Studentized $c = 2$	Studentized $c = 2.5$	Studentized $c = 3$	Cook distance
100	-0.1 (100)	-0.1 (57)	-0.1 (57)	-0.1 (58)	-0.1 (57)	-0.1 (58)	-0.1 (60)	-0.1 (59)
200	-0.1 (100)	-0.0 (57)	-0.0 (57)	-0.0 (58)	-0.0 (57)	-0.0 (58)	-0.0 (59)	-0.0 (58)
500	-0.0 (100)	-0.0 (57)	-0.0 (57)	-0.0 (58)	-0.0 (57)	-0.0 (58)	-0.0 (59)	-0.0 (58)

Table 2: Monte Carlo percent relative bias and Monte Carlo relative efficiency of several estimators

Are these methods satisfactory?

- In the case of symmetric outliers, robust regression and weighted least squares regression after removing outliers, behave very well in terms of bias and efficiency;
- In the case of asymmetric outliers:
  - Robust regression and weighted least squares regression may work well in some scenarios but they tend to breakdown as the sample size increases
  - Why? Because the tuning constant c (e.g., $c = 1.345$) was fixed $\longrightarrow$ not adaptative
- c should be adaptative $\longrightarrow$ c increases as n increases
- At least two criteria: Determine the value of c that minimizes
  - the estimated mean square error of the robust estimator: complex without simplifying assumptions
  - the maximum estimated conditional bias of the robust estimator; Beaumont et al. (2013); Chen et al. (2022)

Influence of a unit

- How measure the influence (or impact) of a unit?
- We measure the influence of $i \in S_r$ (respondent) using the concept of conditional bias:

$$B_i = \mathbb{E}_m \mathbb{E}_p \mathbb{E}_q \left(\widehat{t}_{I,WLS} - t_y \mid Y_i = y_i, I_i = 1, r_i = 1 \right).$$

- After some algebra, we obtain

$$B_i \approx \sum_{j \in U} \left(\frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j} \right) y_j + d_i \left(\sum_{\ell \in U} (1 - p_\ell) \mathrm{x}_\ell^\top \right) \left(\sum_{\ell \in U} p_\ell \mathrm{x}_\ell \phi_\ell^{-1} \mathrm{x}_\ell^\top \right)^{-1} \mathrm{x}_i \phi_i^{-1} (y_i - \mathrm{x}_i^\top \mathrm{B})$$

- First term on the right hand-side: influence of unit i on the sampling error
- Second term on the right hand-side: influence of unit i on the nonresponse error
- B_i: unknown $\longrightarrow$ It must be estimated

Influence of a unit

- Special case: simple random sampling without replacement and simple linear regression imputation (i.e., $\mathrm{x}_i = (1, x_i)^\top$ and $\phi_i = 1$):

$$\widehat{B}_i \approx \left(\frac{N}{n} - 1\right)(y_i - \overline{y}_I) + \frac{1}{\widehat{p}}\left\{(1 - \widehat{p}) + \frac{(x_i - \overline{x})(\overline{x} - \overline{x}_r)}{s_{xr}^2}\right\}\left(y_i - \widehat{B}_{0,WLS} - \widehat{B}_{1,WLS} x_i\right),$$

  where

$$\overline{y}_I = \widehat{t}_I / N, \quad \widehat{p} = n_r / n, \quad s_{xr}^2 = (n_r - 1)^{-1} \sum_{i \in S_r} (x_i - \overline{x}_r)^2$$

- Responding unit i has a large influence if
  - The sampling fraction n/N is small;
  - Its y-value is far from the overall estimated mean $\overline{y}_I$;
  - The response rate is low;
  - Its x-value is far from the overall estimated mean $\overline{x} \longrightarrow$ high leverage point;
  - It has a large vertical residual, $y_i - \widehat{B}_{0,WLS} - \widehat{B}_{1,WLS} x_i$

First proposal

- Following Beaumont et al. (2013), we consider a robust version of

$$\widehat{t}_{I,WLS} = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{WLS}$$

based on the concept of conditional bias:

$$\widehat{t}_{I,CB}(c) \;=\; \widehat{t}_{I,WLS} - \sum_{i \in S_r} \widehat{B}_i + \sum_{i \in S_r} \psi_c \left\{ \widehat{B}_i \right\} \equiv \widehat{t}_{I,WLS} + \Delta(c),$$

where $\psi_c(\cdot)$ denotes the Huber function.

- Proposal: select the value of c that minimizes

$$\max_{i \in S_r} \left| \widehat{B}_i^R \right|,$$

where $\widehat{B}_i^R$ is the conditional bias (influence) of unit i on the robust estimator $\widehat{t}_{I,CB}(c)$.

First proposal

- Resulting estimator:

$$\widehat{t}_{I,CB}(c_{opt}) \;=\; \widehat{t}_{I,WLS} - \frac{1}{2}\left[\min_{i\in S_r}\left\{\widehat{B}_i\right\} + \max_{i\in S_r}\left\{\widehat{B}_i\right\}\right]$$

- The value c_{opt} is obtained by solving

$$\Delta(c) = -\frac{1}{2}\left[\min_{i\in S_r}\left\{\widehat{B}_i\right\} + \max_{i\in S_r}\left\{\widehat{B}_i\right\}\right]$$

- There always exists a solution to the previous equation but the solution may not be unique; see Beaumont et al. (2013) and Favre Martinoz et al. (2015).

- c_{opt} increases as n increases $\longrightarrow$ $\widehat{t}_{I,CB}(c_{opt})$ is a consistent estimator of t_y; see Chen et al. (2022).

Second proposal

- Idea: Propose an adaptative tuning constant c, c_{new}, and use robust regression (based on Huber function say) with this constant.
- Let $\widehat{\mathrm{B}}_{\mathrm{R}}(c_{\text{new}})$ be the solution of

$$\sum_{i \in S_r} \psi_{c_{\text{new}}}\left(\frac{y_i - \mathrm{x}_i^\top \boldsymbol{\beta}}{\widehat{\sigma}\sqrt{\phi_i}}\right)\frac{\mathrm{x}_i}{\sqrt{\phi_i}} = 0,$$

  where $\psi(\cdot)$ is the Huber function.
- Should we use the following estimator?

$$\widehat{t}_{I,R}(c_{\text{new}}) = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(c_{\text{new}})$$

- May not be a good idea because we are only "taking care" of the missing values. However, some respondents may also be influential

Second proposal

- If $\phi_i = \boldsymbol{\lambda}^\top \mathrm{x}_i$, then

$$\widehat{t}_{I,WLS} = \sum_{i \in S} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{WLS}}$$

$\longrightarrow$ Projection form.

- Proposal:

$$\widehat{t}_{I,R}(c_{\mathrm{new}}) = \sum_{i \in S} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(c_{\mathrm{new}}),$$

where

$$c_{\mathrm{new}} = 1.345 \left\{ 1 + \frac{\left| \min_{i \in S_r} \left\{ \widehat{B}_i^* \right\} + \max_{i \in S_r} \left\{ \widehat{B}_i^* \right\} \right|}{2} \right\} + \frac{n}{N} \sqrt{n},$$

where $\widehat{B}_i^*$ denotes the standardized version of $\widehat{B}_i$.

Second proposal

$$c_{\text{new}} = 1.345\left\{1 + \frac{\left|\min_{i\in S_r}\left\{\widehat{B}_i^*\right\} + \max_{i\in S_r}\left\{\widehat{B}_i^*\right\}\right|}{2}\right\} + \frac{n}{N}\sqrt{n}$$

- If n/N small, the second term on the right hand-side is small $\longrightarrow$ we can omit it:
  - If the distribution has symmetric outliers, then c_{new} will be slightly larger than 1.345.
  - If the distribution has asymmetric outliers (say to the right), then c_{new} will be larger than 1.345.
- If n gets larger, then the second term on the right hand-side gets larger and $\widehat{\mathrm{B}}_{\mathrm{R}}(c_{\text{new}})$ get closer and closer to $\widehat{\mathrm{B}}_{\mathrm{WLS}}$

Simulation study: Set-up

10,000 iterations of the following process:

(1) Generate a population of size $N = 1,000$;

Models used to generate the populations:

$$y_i \mid x_i \sim \mathcal{D}(\mu_i; \sigma^2 \phi_i),$$

- $\mu_i = \beta_0 + \beta_1 x_i$ and $\phi_i = x_i$; $\quad x_i \sim Gamma(1, 10)$;
- $\mathcal{D}$: Normal, Lognormal, Pareto, Frechet, Weibull, Student, mixture of normals, mixture of lognormals.

(2) From the population, select a sample of size $n = 50; 100; 200$ according to simple random sampling without replacement.

(3) In each sample: generate nonresponse to the y-variable according to an uniform nonresponse mechanism with probability 50%.

Simulation study: Point estimators

- In each sample, we computed four estimators of t_y:
  - The non-robust estimator:

$$\widehat{t}_{I,WLS} = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{WLS}$$

  - The naive estimator:

$$\widehat{t}_{I,R}(1.345) = \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_R(1.345)$$

  - The robust estimator based on the conditional bias:

$$\widehat{t}_{I,CB}(c_{opt}) = \widehat{t}_{I,WLS} - \frac{1}{2}\left[\min_{i \in S_r}\left\{\widehat{B}_i\right\} + \max_{i \in S_r}\left\{\widehat{B}_i\right\}\right]$$

  - The robust estimator based on c_{new}:

$$\widehat{t}_{I,R}(c_{\mathrm{new}}) = \sum_{i \in S} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(c_{\mathrm{new}})$$

Simulation study: Results

	Point estimator	Normal distribution	Lognormal distribution	Pareto distribution
$n = 50$	$\widehat{t}_{I,WLS}$	-0.3 (100)	-0.1 (100)	-0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-0.4 (101)	-13.5 (73.6)	-8.3 (51)
	$\widehat{t}_{I,CB}(c_{opt})$	-0.8 (100)	-7.2 (77)	-4.9 (56)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-0.2 (101)	-8.7 (73)	-7.0 (38)
$n = 100$	$\widehat{t}_{I,WLS}$	0.0 (100)	-0.5 (100)	-0.0 (100)
	$\widehat{t}_{I,R}(1.345)$	0.0 (102)	-14.6 (101)	-8.6 (59)
	$\widehat{t}_{I,CB}(c_{opt})$	-0.3 (100)	-5.7 (84)	-3.8 (57)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-0.3 (100)	-6.1 (79)	-5.2 (39)
$n = 200$	$\widehat{t}_{I,WLS}$	0.0 (100)	-0.2 (100)	-0.0 (100)
	$\widehat{t}_{I,R}(1.345)$	0.0 (102)	-14.6 (151)	-8.6 (87)
	$\widehat{t}_{I,CB}(c_{opt})$	-0.2 (100)	-3.6 (89)	-2.5 (64)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-0.2 (100)	-2.8 (89)	-3.1 (49)

Table 3: Monte Carlo percent relative bias and relative efficiency of several estimators

Simulation study: Results

	Point estimator	Frechet distribution	Weibull distribution	Student distribution
$n = 50$	$\widehat{t}_{I,WLS}$	-0.1 (100)	0.0 (100)	0.4 (100)
	$\widehat{t}_{I,R}(1.345)$	-9.2 (52)	-17.0 (87)	0.3 (73)
	$\widehat{t}_{I,CB}(c_{opt})$	-5.4 (57)	-8.1 (86)	0.0 (81)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-7.6 (43)	-9.5 (86)	-0.0 (74)
$n = 100$	$\widehat{t}_{I,WLS}$	0.0 (100)	-0.1 (100)	0.0 (100)
	$\widehat{t}_{I,R}(1.345)$	-9.4 (67)	-17.9 (122)	0.1 (72)
	$\widehat{t}_{I,CB}(c_{opt})$	-4.1 (65)	-5.7 (92)	-0.1 (84)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-5.6 (51)	-5.7 (92)	-0.1 (78)
$n = 200$	$\widehat{t}_{I,WLS}$	0.0 (100)	-0.0 (100)	-0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-9.7 (93)	-18.5 (192)	0.0 (71)
	$\widehat{t}_{I,CB}(c_{opt})$	-3.0 (69)	-3.6 (95)	-0.2 (87)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-3.4 (54)	-3.6 (95)	-0.0 (89)

Table 4: Monte Carlo percent relative bias and relative efficiency of several estimators

Simulation study: Results

	Point estimator	Mixture normal (0.01)	Mixture normal (0.03)	Mixture normal (0.05)
$n = 50$	$\widehat{t}_{I,WLS}$	0.1 (100)	-0.1 (100)	-.0.5 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.8 (78)	-5.2 (67)	-7.6 (65)
	$\widehat{t}_{I,CB}(c_{opt})$	-1.8 (83)	-3.8 (79)	-4.5 (82)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-2.2 (76)	-6.0 (71)	-8.0 (79)
$n = 100$	$\widehat{t}_{I,WLS}$	0.1 (100)	-0.1 (100)	0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.9 (78)	-5.3 (72)	-8.1 (78)
	$\widehat{t}_{I,CB}(c_{opt})$	-1.5 (85)	-3.1 (86)	-3.8 (91)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-1.7 (79)	-4.6 (79)	-6.3 (89)
$n = 200$	$\widehat{t}_{I,WLS}$	0.0 (100)	0.1 (100)	-0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.9 (82)	-5.2 (85)	-7.7 (101)
	$\widehat{t}_{I,CB}(c_{opt})$	-1.2 (89)	-2.0 (93)	-2.1 (96)
	$\widehat{t}_{I,R}(c_{\text{new}})$	-0.7 (90)	-2.0 (91)	-1.7 (96)

Table 5: Monte Carlo percent relative bias and relative efficiency of several estimators

Simulation study: Results

	Point estimator	Mixture lognormal (0.01)	Mixture lognormal (0.03)	Mixture lognormal (0.05)
$n = 50$	$\widehat{t}_{I,WLS}$	0.1 (100)	0.0 (100)	-.0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.6 (55)	-4.0 (48)	-6.1 (51)
	$\widehat{t}_{I,CB}(c_{opt})$	-1.3 (63)	-2.8 (63)	-3.9 (69)
	$\widehat{t}_{I,R}(c_{\mathrm{new}})$	-2.0 (44)	-5.4 (47)	-7.9 (61)
$n = 100$	$\widehat{t}_{I,WLS}$	0.0 (100)	0.0 (100)	0.1 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.8 (59)	-4.1 (58)	-5.0 (63)
	$\widehat{t}_{I,CB}(c_{opt})$	-1.2 (66)	-2.4 (72)	-3.1 (80)
	$\widehat{t}_{I,R}(c_{\mathrm{new}})$	-1.8 (48)	-4.7 (57)	-6.8 (79)
$n = 200$	$\widehat{t}_{I,WLS}$	0.0 (100)	-0.1 (100)	0.0 (100)
	$\widehat{t}_{I,R}(1.345)$	-1.8 (66)	-4.0 (79)	-3.6 (81)
	$\widehat{t}_{I,CB}(c_{opt})$	-0.9 (73)	-1.7 (83)	-2.1 (90)
	$\widehat{t}_{I,R}(c_{\mathrm{new}})$	-1.3 (58)	-3.3 (72)	-4.6 (96)

Table 6: Monte Carlo percent relative bias and relative efficiency of several estimators

Implementation via calibrated imputation

- Both robust estimators

$$\widehat{t}_{I,CB}(c_{opt}) = \widehat{t}_{I,WLS} - \frac{1}{2}\left[\min_{i\in S_r}\left\{\widehat{B}_i\right\} + \max_{i\in S_r}\left\{\widehat{B}_i\right\}\right]$$

and

$$\widehat{t}_{I,R}(c_{\text{new}}) = \sum_{i\in S} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}_{\mathrm{R}}(c_{\text{new}})$$

need to be implemented.

- Estimation of totals: data users simply compute

$$\widehat{t}_I = \sum_{i\in S} d_i \widetilde{y}_i, \quad \widetilde{y}_i = r_i y_i + (1 - r_i) y_i^*$$

- How to implement these estimator? $\longrightarrow$ Calibrated imputation

Implementation via calibrated imputation

- Calibrated robust imputation: e.g., Ren and Chambers (2003), Beaumont (2005) and Chen et al. (2022)
- Illustration for $\widehat{t}_{I,R}(c_{\text{new}})$
- Initial imputed values: $y_i^* = \mathrm{x}_i^\top \widehat{\mathrm{B}}_{WLS}$
- We seek final imputed values, $y_{iF}^*,\ i \in S_m$, that minimize

$$\sum_{i \in S} G(y_{iF}^*/y_i^*),$$

subject to

$$\widehat{t}_{I,F} \equiv \sum_{i \in S_r} d_i y_i + \sum_{i \in S_m} d_i y_{iF}^* = \sum_{i \in S} d_i \mathrm{x}_i^\top \widehat{\mathrm{B}}(c_{\text{new}}),$$

where $G(\cdot)$ is a pseudo-distance function.

Estimation of the mean square error

- Estimator of the mean square error of $\widehat{t}_{I,R}(c_{\mathrm{new}})$:

$$\widehat{\mathrm{MSE}} = \widehat{\mathbb{V}}\left(\widehat{t}_{I,R}(c_{\mathrm{new}})\right) + \max\left\{0, \left(\widehat{t}_{I,R}(c_{\mathrm{new}}) - \widehat{t}_{I,WLS}\right)^2 - \widehat{\mathbb{V}}\left(\widehat{t}_{I,R}(c_{\mathrm{new}}) - \widehat{t}_{I,WLS}\right)\right\}$$

- Obtaining the terms $\widehat{\mathbb{V}}\left(\widehat{t}_{I,R}(c_{\mathrm{new}})\right)$ and $\widehat{\mathbb{V}}\left(\widehat{t}_{I,R}(c_{\mathrm{new}}) - \widehat{t}_{I,WLS}\right)$ may be obtained using a pseudo-population bootstrap procedure, motivated by the reverse approach of Shao and Steel (1999) for variance estimation in the presence of imputed data.

- Future work: Conduct a simulation study to assess the performance of $\widehat{\mathrm{MSE}}$, in terms of bias.

THANK YOU.