Value Functions and Fuzzy Measures

Good Practices for Composite Indices Modelling

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The Role of National Statistical Offices in the production of Leading, Composite and Sentiment Indicators

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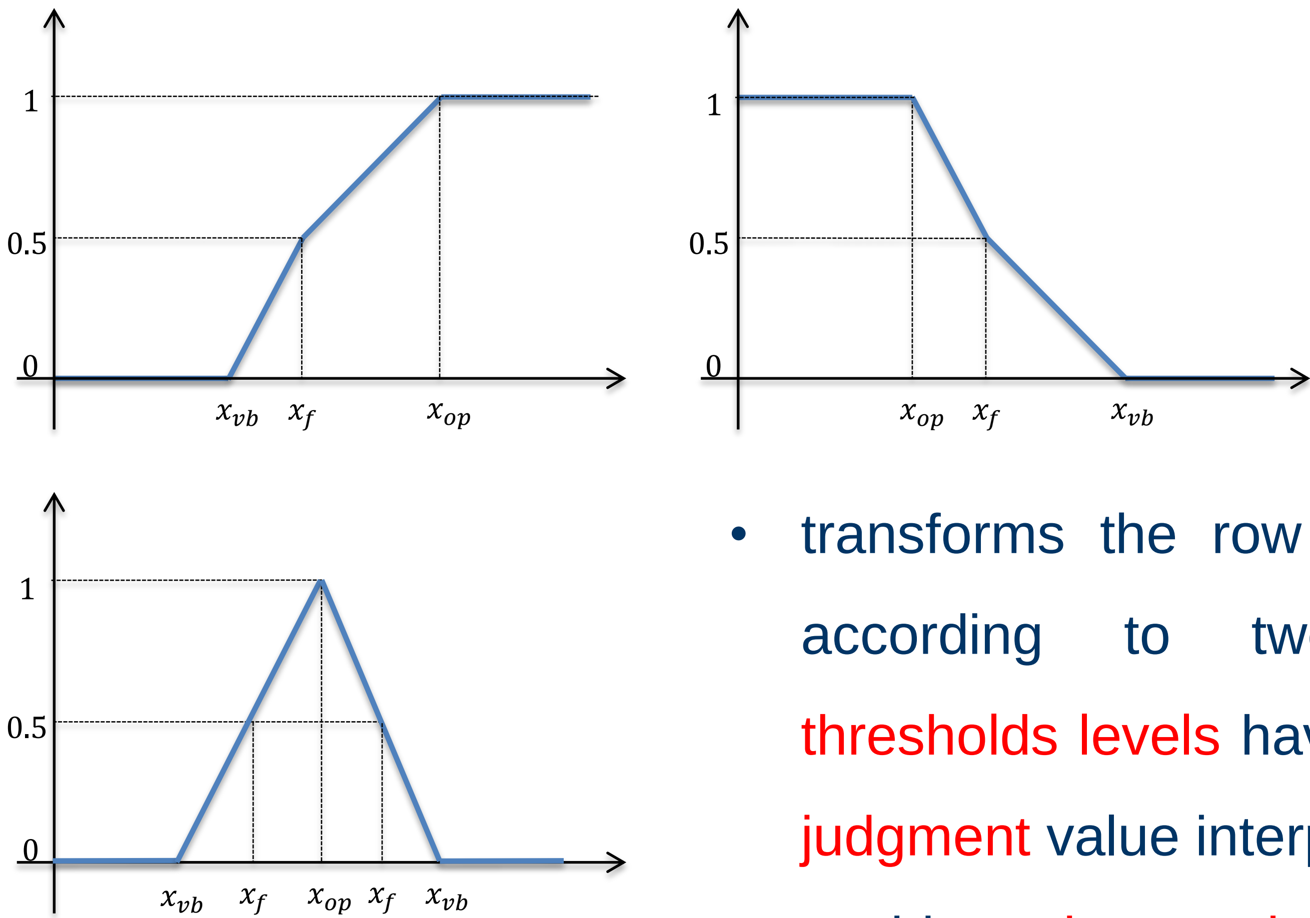
Outline

- Issues on Data Normalization
  - Value Functions
- Weighting Process and Aggregation Function
  - Fuzzy Measures and Choquet Integral
- Application
  - FEEM Sustainability Index
  - Experts' preferences elicitation and weighting
  - Results

Data Normalization

Normalization – Value Functions



- transforms the row criterion data according to two or more thresholds levels having an implicit judgment value interpretation;
- avoids rank reversion problem and conceptual mistakes.

Flawed Data Normalization and Standardization

Experiment

Min-Max Normalization – Max Bias

Crit. A	Crit. B	Score_MM	Rank_MM	Score_True	Rank_True	Abs_Bias
1.8	5.8	0.590	1	0.380	19	18
9.93	5.01	0.585	2	0.747	1	1
9.9	4.91	0.531	3	0.741	3	0
10	4.86	0.510	4	0.743	2	2
9.91	4.86	0.506	5	0.739	4	1
7.47	5.03	0.472	6	0.625	5	1
7.03	5.00	0.435	7	0.602	6	1
4.58	5.18	0.406	8	0.488	16	8
6.36	4.99	0.396	9	0.568	8	1
4.91	5.09	0.376	10	0.500	12	2
5.02	5.07	0.371	11	0.505	11	0
4.88	5.07	0.364	12	0.498	15	3
5.06	5.05	0.362	13	0.506	10	3
7.04	4.84	0.352	14	0.594	7	7
5.30	5.00	0.348	15	0.515	9	6
4.95	5.02	0.341	16	0.499	13	3
4.99	4.98	0.322	17	0.499	14	3
4.84	4.91	0.278	18	0.488	17	1
0.00	5.35	0.266	19	0.268	20	1
4.16	4.9	0.239	20	0.453	18	2

Min-Max Normalization – Max Sum Bias

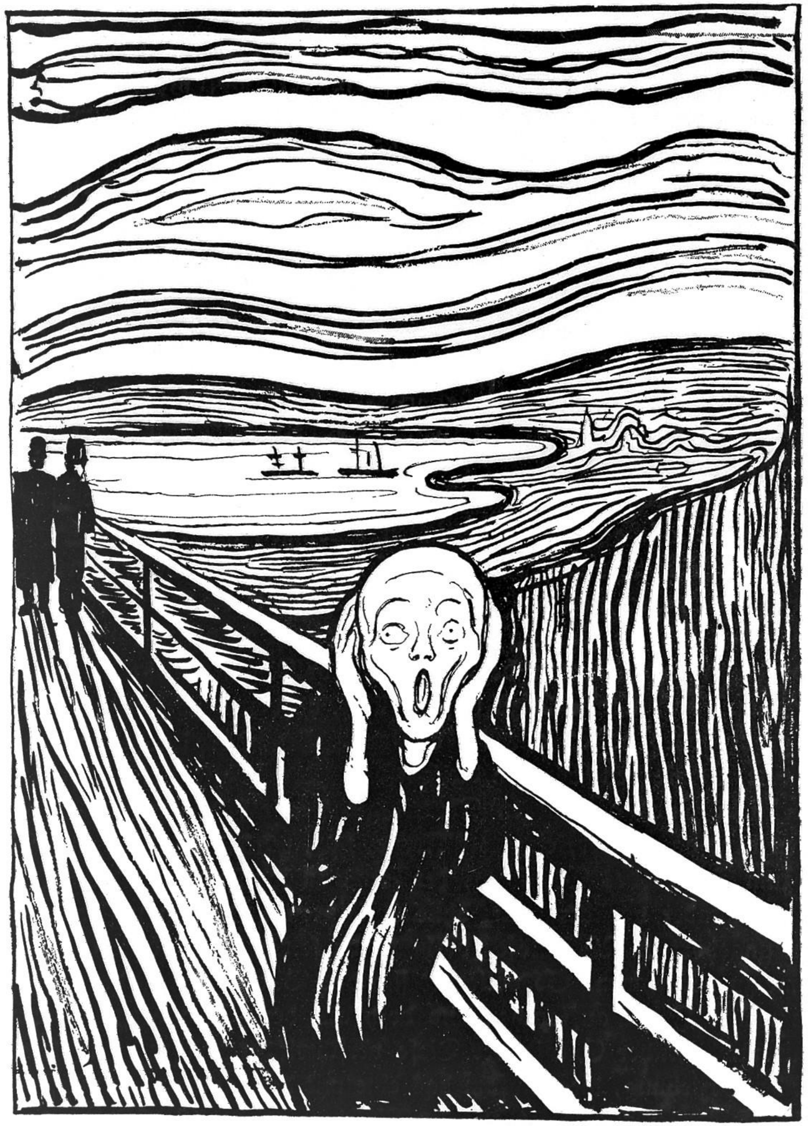
Crit. A	Crit. B	Score_MM	Rank_MM	Score_True	Rank_True	Abs_Bias
6.28	2.72	0.648	1	0.450	17	16
5.96	3.5	0.620	2	0.473	12	10
5.89	3.54	0.606	3	0.472	13	10
5.9	3.12	0.586	4	0.451	16	12
5.56	4.41	0.581	5	0.499	11	6
5.6	3.64	0.548	6	0.462	14	8
5.62	3.52	0.545	7	0.457	15	8
6.27	0.78	0.539	8	0.353	19	11
5.49	3.48	0.514	9	0.449	18	9
4.07	9.06	0.507	10	0.657	1	9
6.29	0	0.500	11	0.315	20	9
4.6	6.76	0.498	12	0.568	4	8
4.04	8.7	0.480	13	0.637	2	11
4.44	7.05	0.478	14	0.575	3	11
4.42	6.91	0.466	15	0.567	5	10
4.48	5.89	0.423	16	0.519	7	9
4.45	5.98	0.421	17	0.522	6	11
4.46	5.9	0.419	18	0.518	8	10
4.42	5.59	0.393	19	0.501	10	9
4.28	6.02	0.386	20	0.515	9	11

Z-Score Standardization – Max Bias

Crit. A	Crit. B	Score_Z	Rank_Z	Score_True	Rank_True	Abs_Bias
0.39	8.19	0.668	1	0.429	20	19
10	4.6	0.574	2	0.730	1	1
9.44	4.77	0.557	3	0.711	2	1
8.9	4.63	0.363	4	0.677	3	1
8.77	4.59	0.312	5	0.668	4	1
4.76	5.93	0.259	6	0.535	7	1
4.5	5.72	0.083	7	0.511	10	3
5.17	5.32	-0.010	8	0.525	9	1
2.42	6.29	-0.017	9	0.436	18	9
6.71	4.67	-0.069	10	0.569	5	5
5.94	4.83	-0.136	11	0.539	6	5
5.6	4.92	-0.154	12	0.526	8	4
4.64	5.2	-0.191	13	0.492	13	0
2.89	5.79	-0.211	14	0.434	19	5
4.39	5.19	-0.249	15	0.479	16	1
5.28	4.8	-0.291	16	0.504	12	4
5.51	4.71	-0.296	17	0.511	11	6
3.7	5.21	-0.381	18	0.446	17	1
4.82	4.81	-0.381	19	0.482	15	4
5.23	4.58	-0.430	20	0.491	14	6

Z-Score Standardization – Max Sum Bias

Crit. A	Crit. B	Score_Z	Rank_Z	Score_True	Rank_True	Abs_Bias
8.83	6.18	0.310	1	0.7505	13	12
7.98	8.03	0.152	2	0.8005	12	10
7.86	8.19	0.118	3	0.8025	11	8
7.5	9.4	0.102	4	0.8450	10	6
10	0	0.096	5	0.5000	17	12
9.97	0	0.083	6	0.4985	18	12
8.9	3.76	0.054	7	0.6330	15	8
8.31	5.87	0.043	8	0.7090	14	6
9.81	0	0.012	9	0.4905	19	10
9.71	0.33	0.007	10	0.5020	16	6
7.12	10	0.005	11	0.8560	1	10
7.09	10	-0.009	12	0.8545	2	10
7.08	10	-0.013	13	0.8540	3	10
7.07	10	-0.017	14	0.8535	4	10
7.06	10	-0.022	15	0.8530	5	10
7.05	10	-0.026	16	0.8525	6	10
7.01	10	-0.044	17	0.8505	7	10
7	10	-0.048	18	0.8500	8	10
6.92	10	-0.084	19	0.8460	9	10
8.16	0	-0.719	20	0.4080	20	0



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Weighting and Aggregation

Weighting and Aggregation

Many composite indices existing in the literature assume implicitly the preferential independence among criteria assumption [D. Scott, P. Suppes, 1958; Fishburn, 1970; Marichal, Roubens 1998]

$\leftrightarrow$ *Arithmetic Mean, Weighted Mean*.

Depending on the objective of the composite indicator, this assumption could be limited or unrealistic because indicators may have some form of synergies and redundancies that should be taken into account in the aggregation phase

$\leftrightarrow$ *Geometric Mean, Constant Elasticity of Substitution*

However these aggregator functions are not flexible enough.

Fuzzy Measures and Choquet Integral

- Allow to model flexible interactions (from redundancy to synergy) and to consider complex dependencies among criteria, hence relaxing the *preferential independence assumption;*

- They can exactly replicate:
  - Weighted mean;
  - Ordered Weighted Averaging Operators (OWA);
  - Minimum and Maximum Operator.

- They are a powerful tool to elicit Experts’ preferences.

Example

To a certain extent:

i) *substitutability between Mathematics and Statistics*

ii) *complementarity between Mathematics and Literature*

iii) *complementarity between Statistics and Literature*

Student	***Math.***	***Stat.***	***Liter.***	***Choquet***
a	10	5	8	8.6
b	5	10	8	8.6
c	8	5	10	7.9
d	10	8	5	7.4
e	8	10	5	7.4
f	6	7	10	7.2

Example – Veto Effect

i) *Mathematics is a veto criterion*

ii) *Statistics $\succ$ Literature*

Student	Math.	Stat.	Liter.	Choquet
a	10	10	10	10
b	10	10	9	9.7
c	10	9	10	9.5
d	7	8	9	7
e	7	10	5	6.4
f	7	5	10	6
g	6	7	7	6
h	6	10	5	5.7
i	6	5	10	5.5
l	5	10	10	5

Fuzzy Measures and Choquet Integral

A Fuzzy measures or Non Additive Measure, defined over the set of criteria $N = \{1,2,\ldots,n\}$, is a set function $\mu: 2^n \rightarrow [0,1]$ satisfying the following boundary and monotonicity conditions:

$$\begin{cases} \mu(\emptyset) = 0 \\ \mu(N) = 1 \\ \mu(S) \leq \mu(T) \leq 1, \qquad \forall\, S \subseteq T \subseteq N \end{cases}$$

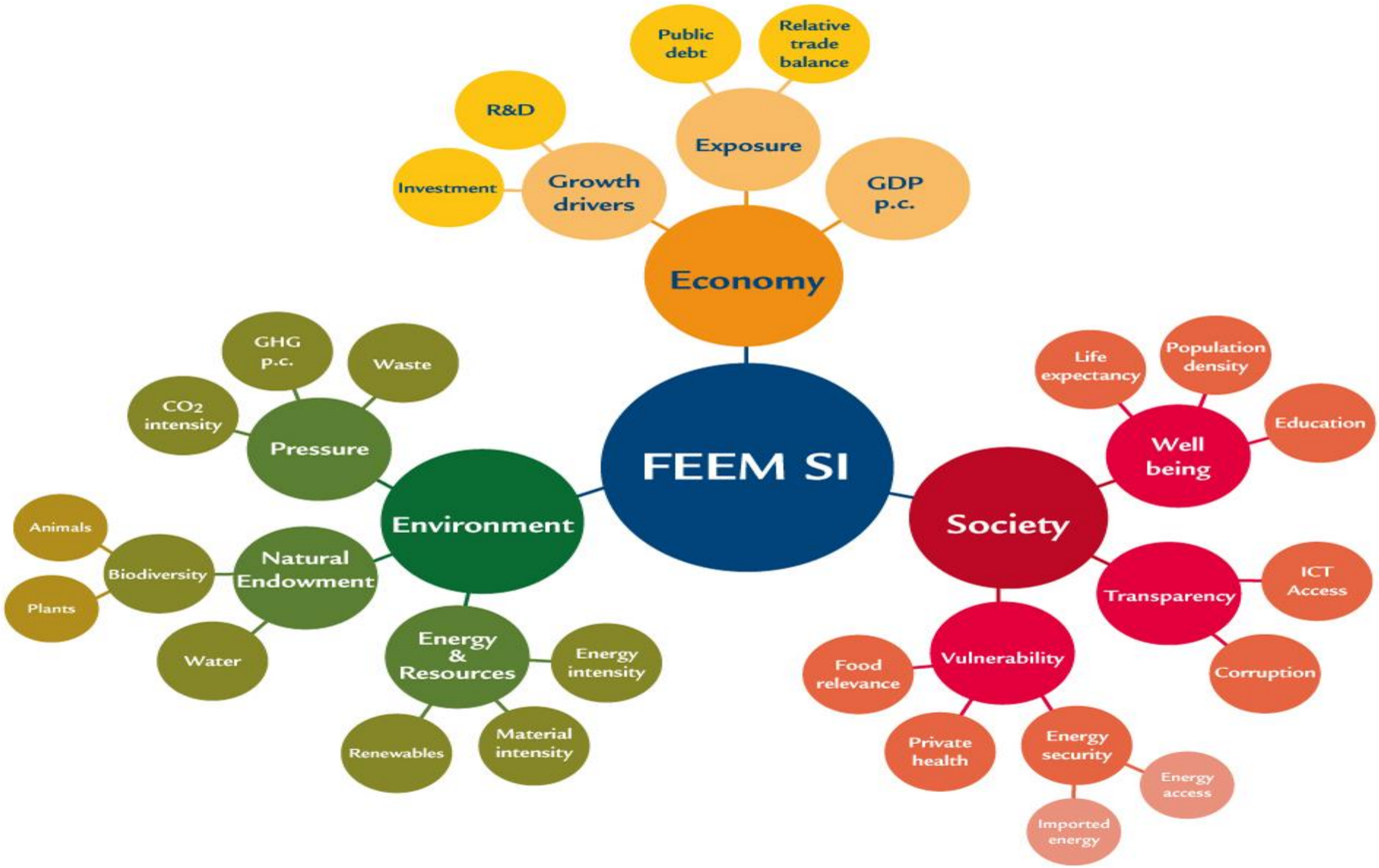
Choquet Integral:

$$Ch(x_1,\ldots,x_n) = \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot \mu\left(A_{(i)}\right)$$

$x_{(0)} \equiv 0$; $x_{(1)} \leq \cdots \leq x_{(n)}$; $A_{(i)} = \left\{x_{(i)},\ldots,x_{(n)}\right\}$

Sustainability Composite Index

FEEM Sustainability Index



L. Farnia, S. Giove (2015), "Fuzzy measures and Experts' opinion elicitation", Smart Innovation, Systems and Technologies

LS Experts' Preference Elicitation

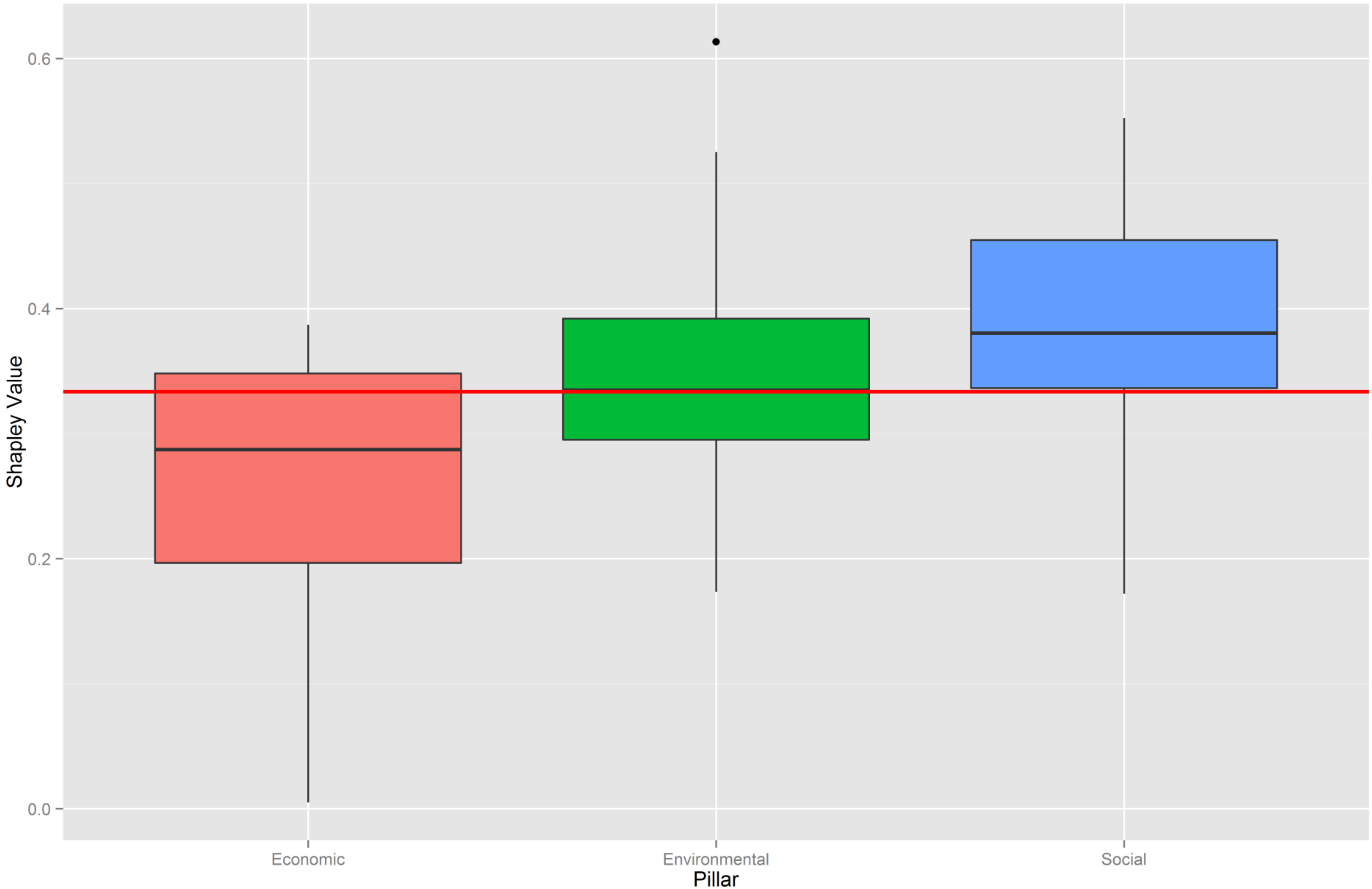
	Criteria			Expert
Alternative	*Env*	*Eco*	*Soc*	*Overall Evaluation*
1	Excellent	Good	Very bad	-
2	Excellent	Very bad	Good	-
3	Good	Excellent	Very bad	-
4	Very bad	Excellent	Good	-
5	Very bad	Good	Excellent	-

$$LS:\ MIN_{m(T)} \sum_{a=1}^{v} (C_m(a) - y(a))^2$$

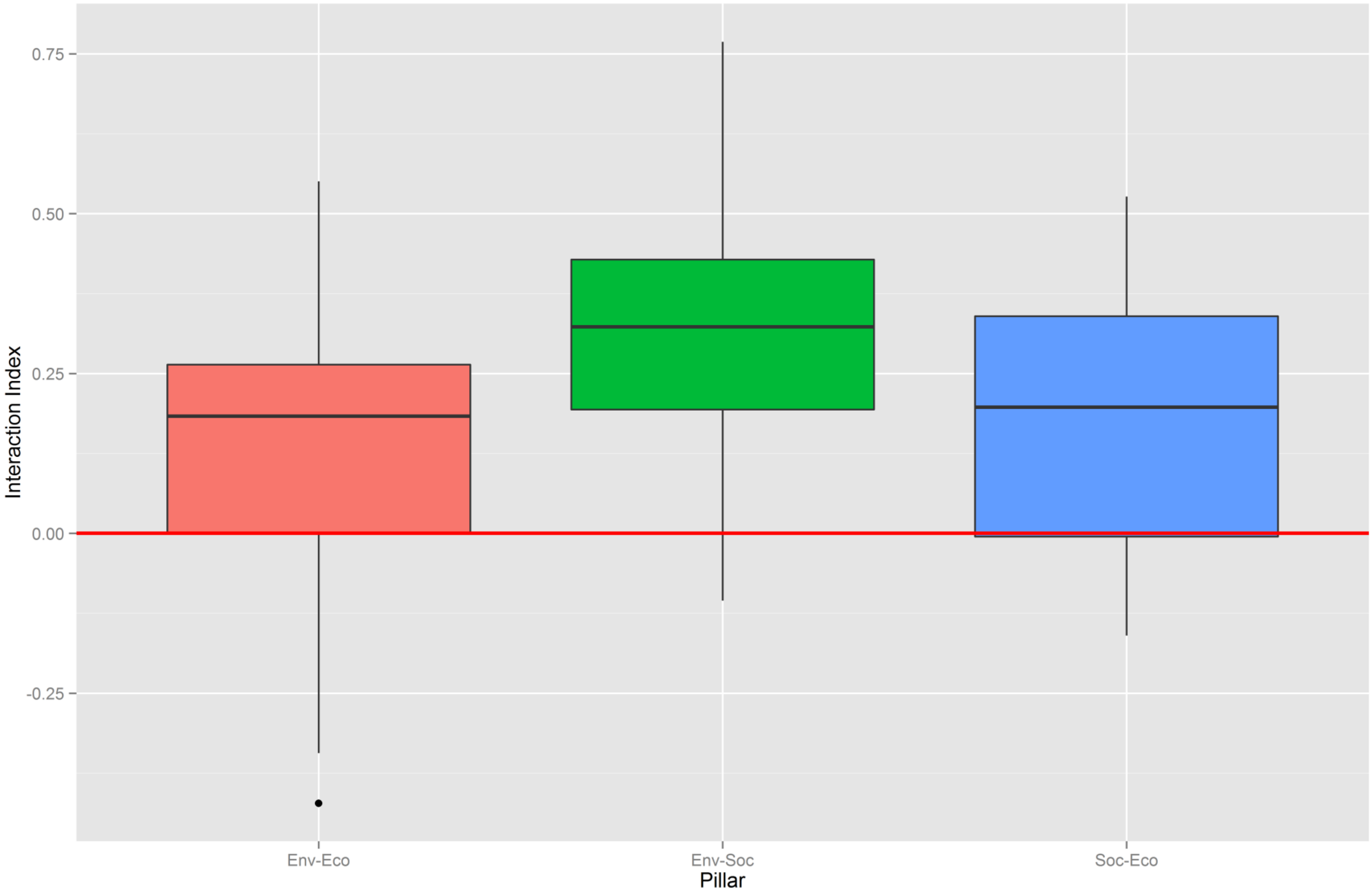
$$s.t.$$

$$\left\{ \begin{array}{l} \sum\limits_{\substack{T \subseteq S \\ T \ni i}} m(T) \geq 0 \ \forall\, S \subseteq N, \forall\, i \in S \\ \sum_{T \subseteq N} m(T) = 1 \end{array} \right\} \quad \textit{Boundary and monotonicity conditions}$$

Shapley Value

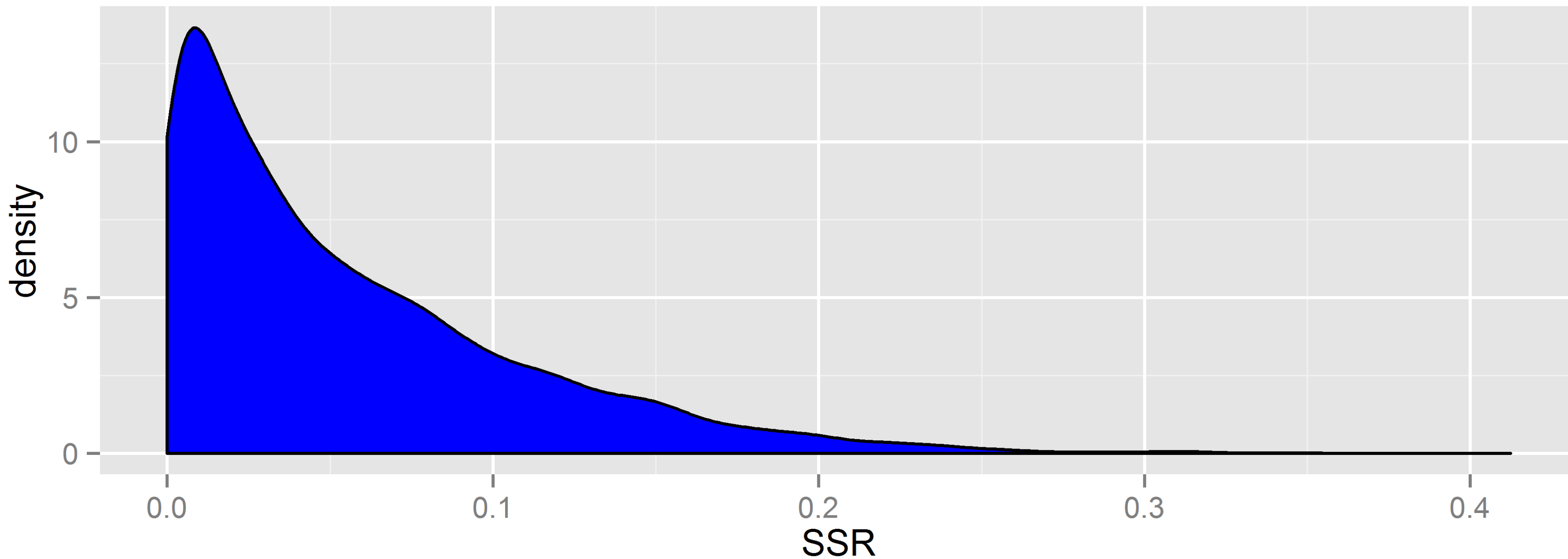


Interaction Index

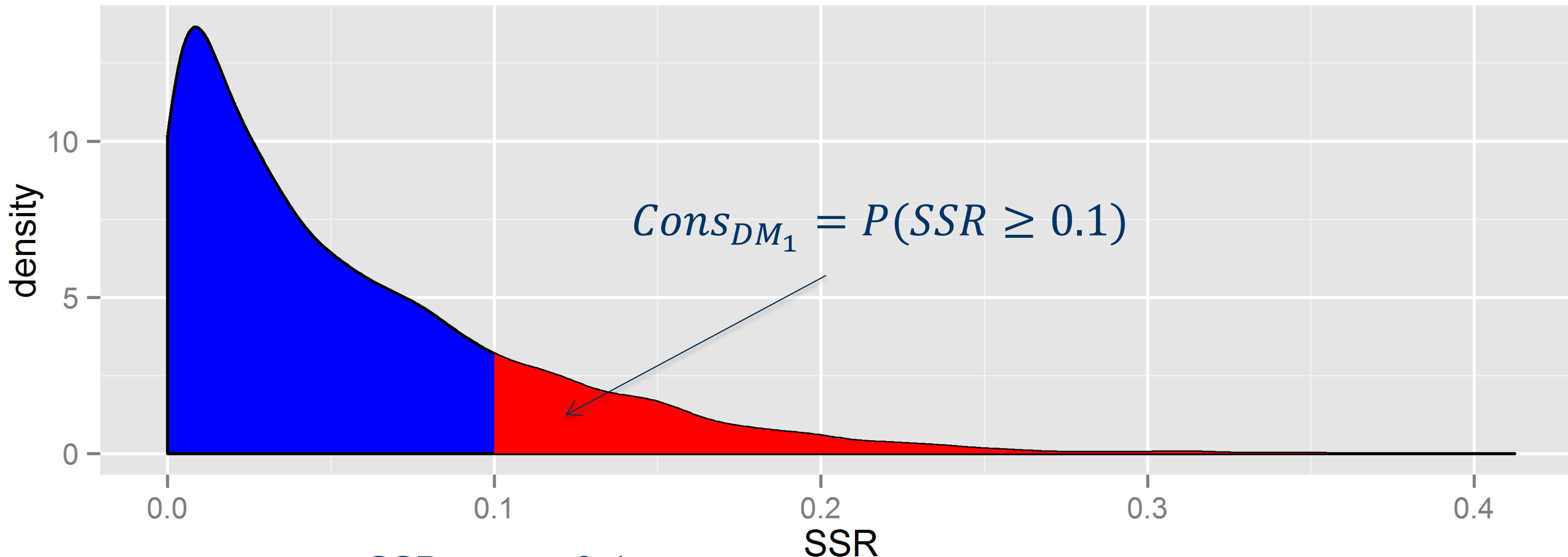


Aggregation of Individual Opinions

Experts' Preferences Weighting Scheme



Experts’ Preferences Weighting Scheme



$$SSR_{DM_1} = 0.1$$

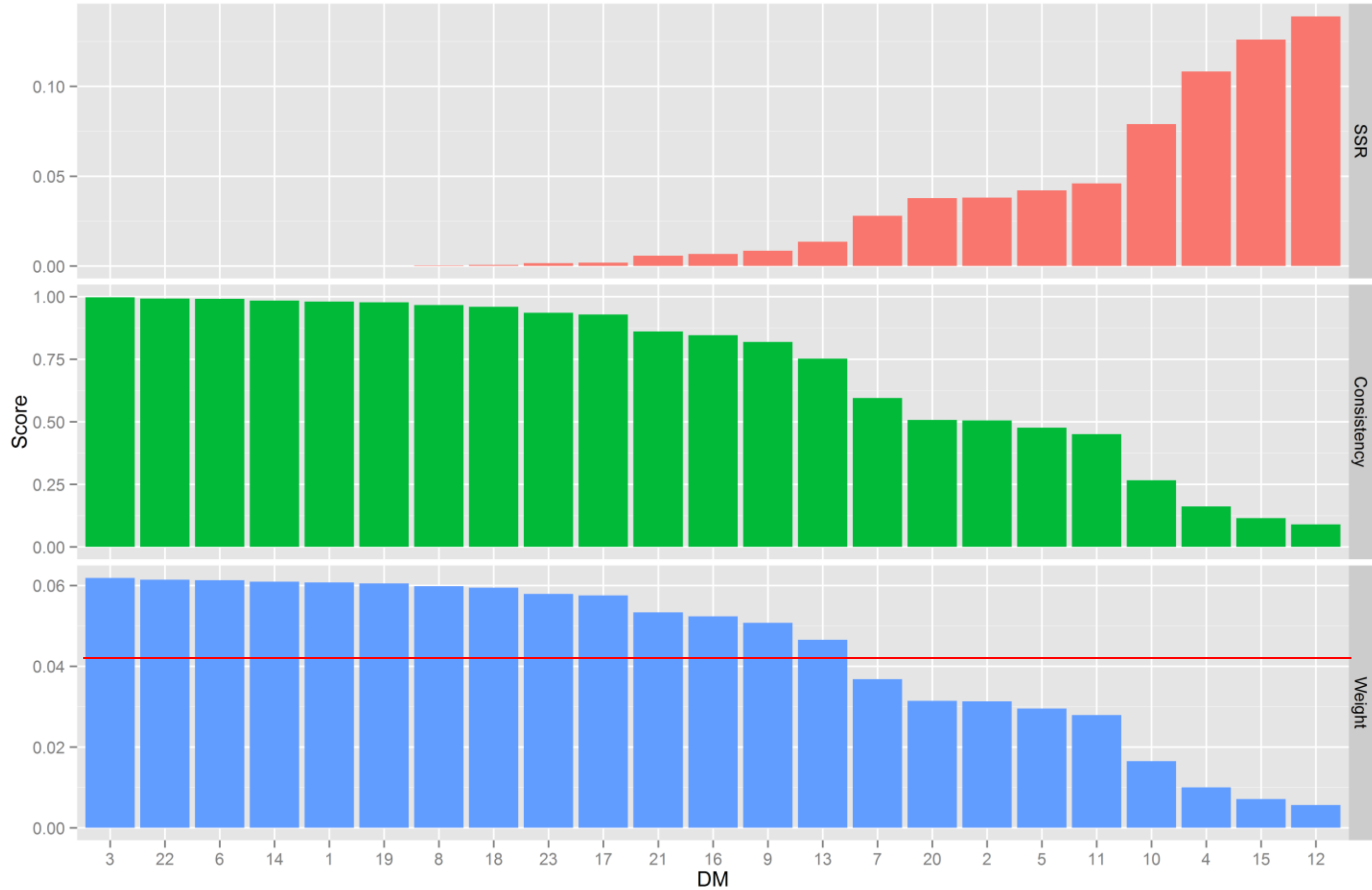
Aggregation of Individual Opinions

$$\mathbf{y}_j = \mathbf{Ch}_j + \boldsymbol{\varepsilon}_j$$

$$SSR_j = \boldsymbol{\varepsilon}_j{}'\boldsymbol{\varepsilon}_j$$

$$Con_j = P\left(SSR \geq SSR_j\right)$$

$$w_j = \frac{Con_j}{\sum_{j=1}^{d} Con_j}$$



$$\Longrightarrow \quad m^*\{T\} = \sum_{j=1}^{d} w_j m_j\{T\} \quad \forall\, T \subseteq N \text{ and } t \leq k$$

Aggregated Preferences

Node	Criteria	Shapley
FEEM SI	Environment	0.357
	Society	***0.386***
	Economy	0.257
Society	Vulnerability	0.295
	Well-Being	***0.412***
	Transparency	0.293
Environment	***Natural Endowment***	***0.356***
	Energy & Resources	0.307
	Pollution	0.337
Economy	***Growth Drivers***	***0.383***
	Exposure	0.314
	GDP p.c.	0.302

Ranking 2013 - Sensitivity

Region	Mean	Min	Max	St.Dev	Best Rank	Worst Rank
Sweden	0.626	0.578	0.676	0.014	1	2
Norway	0.582	0.497	0.679	0.027	1	5
Switzerland	0.579	0.520	0.638	0.017	1	5
Austria	0.559	0.513	0.615	0.014	2	6
Finland	0.539	0.475	0.605	0.019	3	8
France	0.527	0.479	0.575	0.012	4	10
Canada	0.502	0.435	0.568	0.021	6	17
…	…	…	…	…	…	…
EastAsia	0.335	0.276	0.392	0.017	28	37
China	0.328	0.198	0.456	0.038	20	40
Greece	0.319	0.275	0.369	0.015	29	40
RoFSU	0.300	0.225	0.376	0.023	33	40
Indonesia	0.299	0.232	0.352	0.017	31	40
RoAsia	0.290	0.241	0.342	0.015	34	40
India	0.278	0.200	0.345	0.019	37	40

Conclusions

Depending on the objective of the composite indicator, it is important to:

- transform the row criterion data according to two or more thresholds levels having an implicit judgment value interpretation (Value functions);
- “weigh” the considered criteria and to take into account the interactions among them (Fuzzy Measures and Choquet Integral);
- weigh individual opinion according to the consistency of each Decision Maker in answering the proposed questionnaire.

Thank you

Extra Slides

Fuzzy Measures

A Fuzzy measures or Non Additive Measure, defined over the set of criteria $N = \{1,2,\dots,n\}$, is a set function $\mu: 2^N \rightarrow [0,1]$ satisfying the following boundary and monotonicity conditions:

$$\begin{cases} \mu(\emptyset) = 0 \\ \mu(N) = 1 \\ \mu(S) \leq \mu(T) \leq 1, \qquad \forall\, S \subseteq T \subseteq N \end{cases}$$

Example: Two criteria $N = \{1,2\}$

$$\mu\{\emptyset\} = 0;\ \mu\{1,2\} \geq \mu\{1\};\ \mu\{1,2\} \geq \mu\{2\};\ \mu\{1,2\} \equiv 1$$

- If $\mu\{1\} + \mu\{2\} > \mu\{1,2\}$ sub-additive $\rightarrow \{1,2\}$ substitutes
- If $\mu\{1\} + \mu\{2\} < \mu\{1,2\}$ super-additive $\rightarrow \{1,2\}$ complements
- If $\mu\{1\} + \mu\{2\} = \mu\{1,2\}$ additive $\rightarrow \{1,2\}$ independendent

The Möbius representation of Fuzzy Measures allow to reduce the problem complexity (k-additivity); the monotonicity and boundary conditions are:

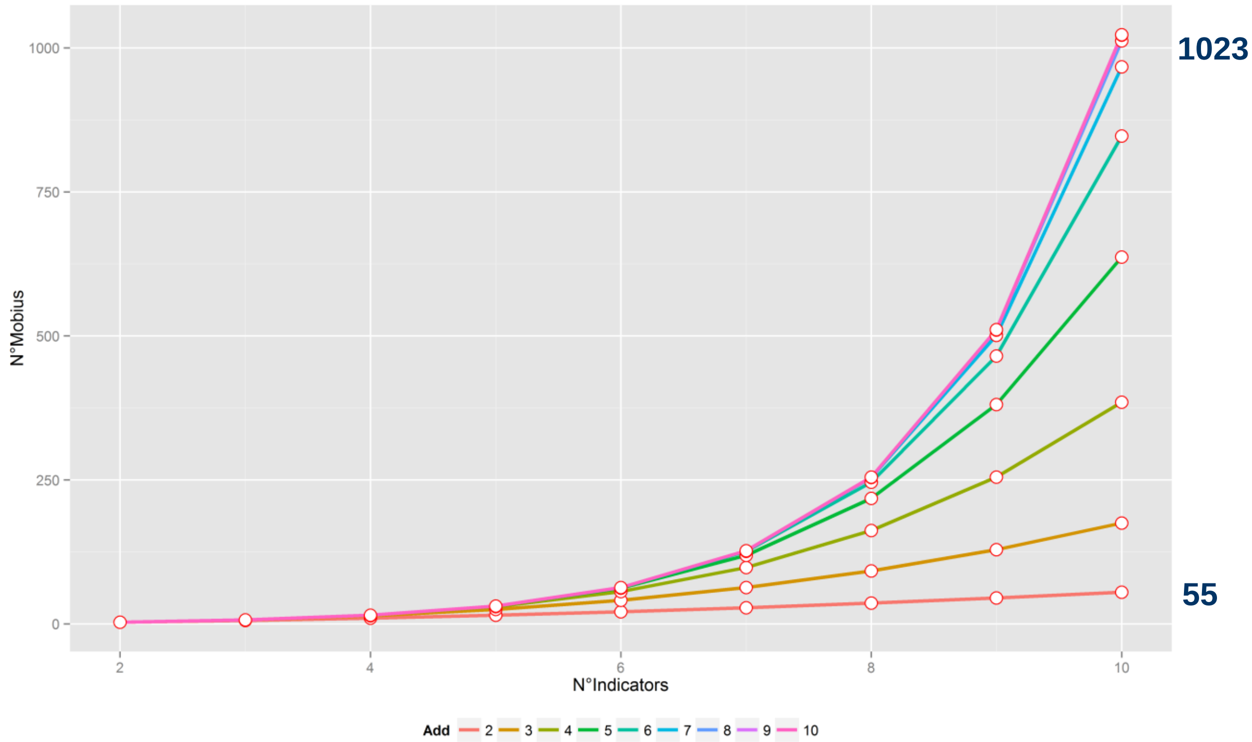
$$\begin{cases} m(\emptyset) = 0 \\ \sum_{T \subseteq N} m(T) = 1 \\ \sum_{\substack{T \subseteq S \\ T \ni i}} m(T) \geq 0, \qquad \forall\, S \subseteq N, \qquad \forall\, i \in S \\ m(T) = 0 \;\; \forall\, t > k \end{cases}$$

The Choquet Integral with respect to Möbius representation:

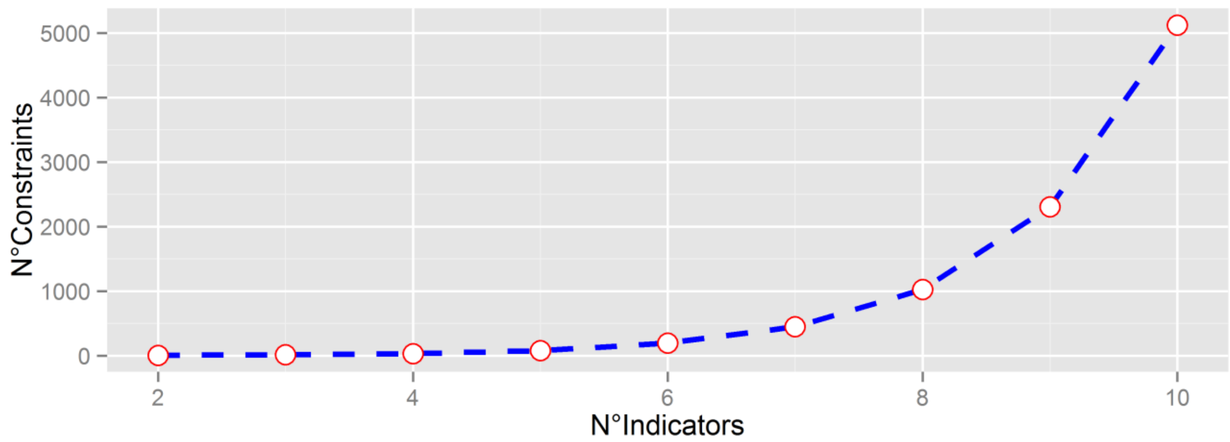
$$C_m(x_1, \ldots, x_n) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} x_i$$

Price to pay

$$N°Möbius = \sum_{j=1}^{k} \binom{n}{j}$$



$$N°Cons. = n \cdot 2^{(n-1)}$$



Social Pillar

Pillar	Node	Criteria	Shapley
Social Pillar	Society	Vulnerability	0.295
		Well-Being	***0.412***
		Transparency	0.293
	Vulnerability	Food Relevance	0.339
		Private Health	0.323
		Energy Security	0.338
	Energy Security	Energy Imported	0.297
		Energy Access	0.703
	Well-Being	Population Density	0.210
		Education	***0.491***
		Life Expectancy	0.299
	Transparency	Corruption	0.705
		ICT Access	0.295

Environmental Pillar

Pillar	Node	Criteria	Shapley
Environmental Pillar	Environment	***Natural Endowment***	***0.356***
		Energy & Resources	0.307
		Pollution	0.337
	Natural Endowment	Water	0.486
		Biodiversity	***0.514***
	Biodiversity	Animals	0.511
		Plants	0.489
	Energy & Resources	Material Intensity	0.320
		Energy Intensity	0.319
		Renewables	0.361
	Pollution	GHG p.c.	0.373
		CO2 Intensity	0.337
		Waste	0.290

Economic Pillar

Pillar	Node	Criteria	Shapley
Economic Pillar	Economy	***Growth Drivers***	***0.383***
		Exposure	0.314
		GDP p.c.	0.302
	Growth Drivers	***R&D***	***0.569***
		Investment	0.431
	Exposure	Rel. Trade Balance	0.577
		Public Debt	0.423