Consumer Price Statistics in the UK

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Abstract

The report makes a number of short and long term recommendations on possible methodological improvements that could be made to the UK's consumer prices statistics by the Office for National Statistics (ONS). The report also reviews several aspects of index number theory summarized in the *Consumer Price Index Manual*, which appeared in 2004 but since then, there have been several important methodological improvements. This new methodology is reviewed and based on this, some additional recommendation are applied to the UK's consumer price statistics. The most important recommendation for improvement is (i) The Retail Price Index (RPI) should drop its use of the Carli index as an elementary index and replace it by either the Jevons or the Carruthers Sellwood Ward and Dalen elementary index.

Key Words

Consumer price indexes, fixed basket indexes, test approaches to index numbers, elementary indexes, the Lowe index, superlative indexes, seasonal commodities, rolling year indexes, scanner data, the GEKS multilateral method, country product dummy and time product dummy methods, fashion goods, the Retail Prices Index (RPI), the Harmonized Index of Consumer Prices (HICP).

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1. Introduction

A main purpose of this report is to assess the suitability of the Retail Prices Index (RPI) and the Consumer Prices Index (CPI) in meeting various purposes for which measures of consumer price inflation are generally used. The Office for National Statistics (ONS) in the UK produces both the RPI and the CPI on a monthly frequency.[2] Another main purpose of this report will be to identify any weaknesses in either of these indexes and to make suggestions as to how these weaknesses might be addressed, both in the short run as well as in the longer run. A brief outline of this report follows.

In section 2 below, some important uses of measures of household price inflation will be discussed. The suitability of the RPI and the CPI will be discussed in the light of these uses after their methodologies have been explained.

At the outset, it should be recognized that no consumer price index will be conceptually perfect: data limitations and cost considerations will prevent the "perfect" index from being produced. However, it will be useful to introduce the various idealized types of index that have been suggested in the literature on index number theory over the past 200 years. These idealized indexes are called *target indexes*. There are four main approaches to the determination of the functional form for a target price index that compares the prices (and associated quantities) between two periods:

- Fixed basket and averages of fixed basket approaches;
- The test or axiomatic approach;
- The stochastic approach and
- The economic approach.

These four approaches will be explained in section 3 below.[3]

Practical consumer price indexes are constructed in two stages:

- A first stage at the lowest level of aggregation where price information is available but associated expenditure or quantity information is not available and
- A second stage of aggregation where expenditure information is available at a higher level of aggregation.

The aggregates that pertain to the first stage of aggregation are called *elementary aggregates*. Again, theories for "ideal" or "best" target indexes can be developed in this

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[2] Since the UK belongs to the European Union, the ONS is legally obligated to produce the European Union's standard measure of household inflation, the Harmonized Index of Consumer Prices or HICP. The CPI that is produced by the ONS is actually the HICP and so there will be some discussion about the methodology used for the HICP in this report. However, since the ONS has little influence on the construction of the HICP, our focus will be on the RPI.

[3] The material in this section largely parallels the material on index number theory that is laid out in the *Consumer Price Index Manual*; see the ILO/IMF/OECD/UNECE/Eurostat/The World Bank (2004; 263-327). For brevity, in the future, we will refer to the *CPI Manual* as ILO (2004).

situation where price information is available but not quantity or expenditure information. The two approaches that have been developed in this context are:

- The test approach and
- The stochastic approach.

Thus the theories for the target index in the elementary aggregate context parallel the theories developed when both price and quantity information is available, except that the fixed basket and economic approaches cannot be applied in the elementary index context. The remaining two approaches to the construction of elementary indexes (the test and stochastic approaches) will be discussed in section 4 below.[4]

All of the above theories for target indexes apply to situations where only the prices of two months are being compared and the comparison formulae do not depend on the prices and quantities of any other month. This is termed bilateral index number theory since only two situations are being compared. Unfortunately, practical index number theory is more complicated: in practical indexes like the RPI and the CPI, an *annual expenditure basket* pertaining to a past year is used at higher levels of aggregation and at the elementary level of aggregation; the prices of twelve consecutive months are compared with the corresponding prices in December or January. The resulting price index is known as a *Lowe index* in the literature and in section 5, we will give an overview of this methodology.

In section 6, the problem of strongly seasonal commodities is addressed in the context of the Lowe index methodology. A *strongly seasonal commodity* is one that is available for only certain months of the year; e.g., Christmas trees, winter clothing and summer clothing.[5] The C*onsumer Price Index Manual* recommended the use of maximum overlap superlative indexes in the context of producing useful month to month consumer price indexes with seasonal commodities. But the evidence in Feenstra and Shapiro (2003), Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011) shows that these maximum overlap indexes can be subject to a tremendous chain drift problem. Thus in section 7, a newer methodological approach to the production of month to month indexes is suggested that avoids the chain drift problem.

The new method for constructing month to month indexes that avoid the chain drift problem is due to Balk (1981), Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011) and is known as the Rolling Year GEKS method. It is described in section 7 along with a useful approximation to this method.

In section 8, a new method for constructing elementary indexes is suggested: the *Rolling Year Time Product Dummy method* (RYTPD method). This is a stochastic approach to

[4] Most of the material in section 4 is also presented in the ILO (2004; 355-371). However, sections 4.5, 4.7 and 4.8 present some new material.

[5] Much of the material in this section is also presented in the ILO (2004; 393-417). However, since this *Consumer Price Index Manual* material was written, some new evidence on seasonal indexes has become available due to Diewert, Finkel and Artsev (2009) and this new material is reviewed in section 6.

elementary indexes that is an adaptation of a method suggested by Summers (1973). It is similar in spirit to the Rolling Year GEKS method, except that the Rolling Year TPD method uses only price information instead of both price and quantity information.

In section 9, clothing and other fashion goods are discussed.

Finally, section 10 concludes with some recommendations for improving the RPI and CPI in the UK.

2. The Scope and Purpose of Consumer Price Indexes

There are *three main purposes* for which it is desirable to measure the average rate of change in consumer prices going from a base period 0 to a comparison period 1 for a well defined household or group of households that pertains to a well specified value aggregate (that defines which commodity transactions are *in scope* or to be included in the value aggregate):

- As a *summary measure of the overall rate of price change* that the specified group faces over the two periods being compared for the value aggregate under consideration. Households in the specified group will generally be interested in this summary number as will governments and central bankers.
- As a *deflator* for the value aggregate under consideration. Deflating the value ratio by the price index gives us the rate of growth of the corresponding quantity aggregate and this rate of growth can often be given a welfare change interpretation. Taking the household sector as a whole, the System of National Accounts requires a deflator for household expenditures.
- As a *compensation or indexation measure*. Governments or private employers may want to index benefit or salary levels to ensure that the indexed entitlement or salary level in the current period is "equivalent" in some sense to the base level of entitlement or salary. The relevant value aggregates for this purpose should be based on the expenditures of the recipients of entitlements or of the employees in the indexation group.

It can be seen that the first two uses are really complimentary aspects of the same problem, which is to decompose a ratio of value aggregates for two periods into a price change component times a quantity change component. However, constructing an index for the third purpose is more difficult, since a proportion of an entitlement or a salary can be saved rather than spent on consumer goods and services and so the determination of the relevant value aggregate is not so easy. Another significant problem is that alternative treatments of purchases of consumer durables can lead to very different entitlement indexes.

Restricting attention to uses 1 and 2 above, it can be seen that the index number problem in dealing with demographic groups (e.g., pensioners) or with a restricted commodity classification (e.g., expenditures are restricted to an agreed on universe of "essential" commodities) is more or less the same as dealing with the entire household sector but the practical problem facing index number producers is the lack of data (or more accurately, the higher costs of collecting detailed expenditure and price data for demographic groups of households).

In practice, the problems facing statistical agencies producing consumer price indexes are more complicated than indicated above. With respect to the first two uses listed above, the theoretical approaches to index number theory that will be explained in section 3 apply to situations where complete price and quantity information on the same N goods and services in the household aggregate is available for the two periods being compared.

Unfortunately, this idealized situation does not apply to the real world for a number of reasons:

- The existence of *strongly seasonal commodities*; i.e., these are commodities such as Christmas trees and swim suits that are available for only certain months of the year.
- The introduction of *new products* and the disappearance of older products that have been rendered *obsolete* by technical progress. This means that the list of comparable commodities changes from month to month.
- *Product churn* and temporary shortages of stock. Many retailers rotate their choice of brands that they will stock on their shelves for various reasons. Again, this leads to a lack of comparability of products in the aggregate across months.
- The *existence of durable goods*. Thus a particular household may purchase a consumer durable (such as a motor vehicle or a house) in the base period but not purchase it in the following period. However, the household will still enjoy the services of the previously purchased consumer durable in the comparison period.

All of the above problems lead to *a lack of matching of purchases* of products across the two periods being compared and this creates problems for all approaches to index number theory. We will attempt to deal with the above problems in subsequent sections.

Problems associated with the HICP and the RPI and recommendations for improving these indexes will be deferred until section 10; i.e., it is first necessary to review possible techniques that have been suggested in the index number literature to improve price measurement.

We turn now to the problem of choosing an explicit index number formula. In the following section, we outline the four main approaches that are in use today to justify various functional forms for the price index when complete price and quantity data for the value aggregate are available for a number of periods. When the comparison of price changes is restricted to two periods, the price index that constructs an average measure of price change is called a bilateral index number formula. In the following section, attention is restricted to two period comparisons but of course, if one period is held fixed (called the base period), then a bilateral price index formula can be used to make a sequence of price comparisons over a number of subsequent periods.

3. Alternative Approaches to Bilateral Index Number Theory

3.1 Setting the Stage

It will be useful to set the stage for the subsequent discussion of alternative approaches by defining more precisely what the index number problem is.

We specify two accounting periods, $t = 0,1$ for which we have micro price and quantity data for N commodities pertaining to transactions by a consumer (or a well defined group of consumers). Denote the price and quantity of commodity n in period t by p_n^t and q_n^t respectively for $n = 1,2,\ldots,N$ and $t = 0,1$. Before proceeding further, we need to discuss the exact meaning of the microeconomic prices and quantities if there are *multiple* transactions for say commodity n within period t. In this case, it is natural to interpret q_n^t as the *total* amount of commodity n transacted within period t. In order to conserve the value of transactions, it is necessary that p_n^t be defined as a *unit value* [6]; i.e., p_n^t must be equal to the value of transactions for commodity n during period t divided by the total quantity transacted, q_n^t. For $t = 0,1$, define *the value of transactions in period t* as:

(1) $V^t \equiv \sum_{n=1}^N p_n^t\, q_n^t \equiv p^t \cdot q^t$

where $p^t \equiv (p_1^t,\ldots, p_N^t)$ is the period t price vector, $q^t \equiv (q_1^t,\ldots, q_N^t)$ is the period t quantity vector and $p^t \cdot q^t$ denotes the inner product of these two vectors.

Using the above notation, we can now state the following *levels version of the index number problem using the test or axiomatic approach*: for $t = 0,1$, find scalar numbers P^t and Q^t such that

(2) $V^t = P^t Q^t$.

The number P^t is interpreted as an aggregate period t price level while the number Q^t is interpreted as an aggregate period t quantity level. The aggregate price level P^t is allowed to be a function of the period t price vector, p^t while the aggregate period t quantity level Q^t is allowed to be a function of the period t quantity vector, q^t; i.e., we have

(3) $P^t = c(p^t)$ and $Q^t = f(q^t)$; $t = 0,1$.

However, from the viewpoint of the *test approach* to index number theory, the levels approach to finding aggregate quantities and prices comes to an abrupt halt: Eichhorn (1978; 144) showed that if the number of commodities N in the aggregate is equal to or greater than 2 and we restrict $c(p^t)$ and $f(q^t)$ to be positive if the micro prices and

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[6] The early index number theorists Walsh (1901; 96), Fisher (1922; 318) and Davies (1924; 96) all suggested unit values as the prices that should be inserted into an index number formula. This advice is followed in the *Consumer Price Index Manual: Theory and Practice* with the proviso that the unit value be a narrowly defined one; see the ILO (2004; 356).

quantities p_n^t and q_n^t are positive, then there do not exist any functions c and f such that $c(p^t)f(q^t) = p^t{\cdot}q^t$ for all $p^t >> 0_N$ and $q^t >> 0_N$.[7]

This negative result can be reversed if we take the *economic approach* to index number theory. In this approach, we assume that the economic agent has a linearly homogeneous utility function, f(q), and when facing the prices p^t chooses q^t to solve the following cost minimization problem:

$$(4)\ \min_q \{p^t{\cdot}q : p^t{\cdot}q = Y^t ;\ q \geq 0_N\};\ t = 0,1$$

where period t "income" Y^t is defined as $p^t{\cdot}q^t$. In this setup, it turns out that c(p) is the unit cost function that is dual[8] to the linearly homogeneous utility function f(q) and we can define P^t and Q^t as in (3) with $P^tQ^t = c(p^t)f(q^t) = p^t{\cdot}q^t$ for t = 0,1. Why does the economic approach work in the levels version of the index number problem whereas the test approach does not? In the test approach, both p^t and q^t are regarded as completely independent variables, whereas in the economic approach, p^t can vary independently but q^t cannot vary independently; it is a solution to the period t cost minimization problem (4).

Even though the economic approach to the index number problem as formulated above "works", it is not a *practical* solution that statistical agencies can implement and provide suitable aggregates to the public. In order to implement this solution, the statistical agency would have to hire hundreds of econometricians in order to estimate cost functions for all relevant macroeconomic aggregates and it is simply not feasible to do this. Thus we turn to our second formulation of the index number problem and it is this formulation that was initiated by Fisher (1911) (1922) in his two books on index number theory.

In the second approach to index number theory, instead of trying to decompose the value of the aggregate into price and quantity components for a single period, we instead attempt to decompose a *value ratio* for the two periods under consideration into a *price change component* P times a *quantity change component* Q.[9] Thus we now look for two functions of 4N variables, $P(p^0,p^1,q^0,q^1)$ and $Q(p^0,p^1,q^0,q^1)$ such that:[10]

$$(5)\ p^1{\cdot}q^1/p^0{\cdot}q^0 = P(p^0,p^1,q^0,q^1)Q(p^0,p^1,q^0,q^1).$$

If we take the test approach, then we want equation (5) to hold for all positive price and quantity vectors pertaining to the two periods under consideration, p^0,p^1,q^0,q^1. If we take

[7] Notation: $p >> 0_N$ means all components of p are positive; $p \geq 0_N$ means all components of p are nonnegative and $p > 0_N$ means $p \geq 0_N$ but $p \neq 0_N$. Finally, $p{\cdot}q \equiv \sum_{n=1}^N p_nq_n$.
[8] See Diewert (1974) for materials and references to the literature on duality theory.
[9] Looking ahead to the economic approach, P will be interpreted to be the ratio of unit cost functions, $c(p^1)/c(p^0)$, and Q will be interpreted to be the utility ratio, $f(q^1)/f(q^0)$. Note that the linear homogeneity assumption on the utility function f effectively cardinalizes utility.
[10] If N = 1, then we define $P(p_1^0,p_1^1,q_1^0,q_1^1) \equiv p_1^1/p_1^0$ and $Q(p_1^0,p_1^1,q_1^0,q_1^1) \equiv q_1^1/q_1^0$, the single price ratio and the single quantity ratio respectively. In the case of a general N, we think of $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as being a weighted average of the price ratios p_1^1/p_1^0, p_2^1/p_2^0, ..., p_N^1/p_N^0. Thus we interpret $P(p_1^0,p_1^1,q_1^0,q_1^1)$ as an aggregate price ratio, P^1/P^0, where P^t is the aggregate price level for period t for t = 0,1.

the economic approach, then only the price vectors p^0 and p^1 are regarded as independent variables while the quantity vectors, q^0 and q^1, are regarded as dependent variables.

In this second approach to index number theory, the *price index* $P(p^0,p^1,q^0,q^1)$ and the *quantity index* $Q(p^0,p^1,q^0,q^1)$ cannot be determined independently; i.e., if either one of these two functions is determined, then the remaining function is implicitly determined using equation (5). Historically, the focus has been on the determination of the price index but Fisher (1911; 388) was the first to realize that once the price index was determined, then equation (5) could be used to determine the companion quantity index.[11] This value ratio decomposition approach to index number is called *bilateral index number theory* and its focus is the determination of "reasonable" functional forms for P and Q. Fisher's 1911 and 1922 books address this functional form issue using the test approach.

We turn now to a discussion of the various approaches that have been used to determine the functional form for the bilateral price index, $P(p^0,p^1,q^0,q^1)$.

3.2 Fixed Basket Approaches to Bilateral Index Number Theory

A very simple approach to the determination of a price index over a group of commodities is the *fixed basket approach*. In this approach, we are given a basket of commodities that is represented by the positive quantity vector q. Given the price vectors for periods 0 and 1, p^0 and p^1 respectively, we can calculate the cost of purchasing this same basket in the two periods, $p^0 \cdot q$ and $p^1 \cdot q$. Then the ratio of these costs is a very reasonable indicator of pure price change over the two periods under consideration, provided that the basket vector q is "representative". Thus define the *Lowe* (1823) *price index*, P_{Lo}, as follows:

(6) $P_{Lo}(p^0,p^1,q) \equiv p^1 \cdot q / p^0 \cdot q$.

As time passed, economists and price statisticians demanded a bit more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the period 0 commodity vector q^0 or the period 1 commodity vector q^1. These two choices lead to the Laspeyres (1871) price index P_L defined by (7) and the Paasche (1874) price index P_P defined by (8):[12]

(7) $P_L(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^0 / p^0 \cdot q^0 = \sum_{n=1}^N s_n^0 (p_n^1/p_n^0)$;

(8) $P_P(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1 / p^0 \cdot q^1 = [\sum_{n=1}^N s_n^1 (p_n^1/p_n^0)^{-1}]^{-1}$

[11] This approach to index number theory is due to Fisher (1911; 418) who called the implicitly determined Q, the *correlative formula*. Frisch (1930; 399) later called (5) the *product test*.
[12] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. However, it does no harm to include these vectors and the notation indicates that we are in the realm of bilateral index number theory.

where the period t expenditure share on commodity n, s_n^t, is defined as $p_n^t q_n^t / p^t \cdot q^t$ for n = 1,…,N and t = 0,1. Thus the Laspeyres price index P_L can be written as a base period expenditure share weighted average of the N price ratios (or price relatives), p_n^1/p_n^0.[13] The last equation in (8) shows that the Paasche price index P_P can be written as a period 1 (or current period) expenditure share weighted *harmonic* average of the N price ratios.[14]

The problem with these index number formulae is that they are equally plausible but in general, they will give different answers. This suggests that if we require a single estimate for the price change between the two periods, then we should take some sort of evenly weighted average of the two indexes as our final estimate of price change between periods 0 and 1. Examples of such symmetric averages are the arithmetic mean, which leads to the Drobisch (1871) Sidgwick (1883; 68) Bowley (1901; 227)[15] index, $(1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the *Fisher* (1922) *ideal index*, P_F, defined as

$$(9)\ P_F(p^0,p^1,q^0,q^1) \equiv [P_L(p^0,p^1,q^0,q^1)\ P_P(p^0,p^1,q^0,q^1)]^{1/2} .$$

At this point, the fixed basket approach to index number theory has to draw on the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indexes or which averages of them might be "best", we need *criteria* or *tests* or *properties* that we would like our indexes to satisfy.

What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test.*[16] We say that the index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$(10)\ \ P(p^1,p^0,q^1,q^0) = 1/P(p^0,p^1,q^0,q^1)\ \ ;$$

i.e., if we interchange the period 0 and period 1 price and quantity data and evaluate the index, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$.

Diewert (1997; 138) showed that the Fisher ideal price index defined by (9) above is the *only* index that is a homogeneous symmetric mean of the Laspeyres and Paasche price indexes, P_L and P_P, and satisfies the time reversal test (10) above. Thus our first

[13] This result is due to Walsh (1901; 428 and 539).
[14] This expenditure share and price ratio representation of the Paasche index is described by Walsh (1901; 428) and derived explicitly by Fisher (1911; 365).
[15] See Diewert (1992) (1993) and Balk (2008) for additional references to the early history of index number theory.
[16] The concept of this test is due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test (and the commensurability test to be discussed later) that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 324) and Fisher (1922; 64).

symmetric basket approach to bilateral index number theory leads to the Fisher index (9) as being "best" from the perspective of this approach.[17]

Instead of looking for a "best" average of the two fixed basket indexes that correspond to the baskets chosen in either of the two periods being compared, we could instead look for a "best" average basket of the two baskets represented by the vectors q^0 and q^1 and then use this average basket to compare the price levels of periods 0 and 1.[18] Thus we ask that the nth quantity weight, q_n, be an average or *mean* of the base period quantity q_n^0 and the period 1 quantity for commodity n q_n^1, say $m(q_n^0,q_n^1)$, for n = 1,2,…,N.[19] Price statisticians refer to this type of index as a *pure price index* and it corresponds to Knibbs' (1924; 43) *unequivocal price index*. Under these assumptions, the pure price index can be defined as a member of the following class of index numbers:

$$(11)\quad P_K(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1 m(q_n^0,q_n^1) / \sum_{j=1}^N p_j^0 m(q_j^0,q_j^1).$$

In order to determine the functional form for the mean function m, it is necessary to impose some *tests* or *axioms* on the pure price index defined by (11). Suppose that we impose the time reversal test (10) and the following *invariance to proportional changes in current quantities test*:

$$(12)\quad P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1) \text{ for all } \lambda > 0.$$

Diewert (2001; 207) showed that these two tests determine the precise functional form for the pure price index P_K defined by (11) above: the pure price index P_K must be the *Walsh* (1901; 398) (1921a; 97) *price index*, P_W[20] defined by (13):

$$(13)\quad P_W(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N p_n^1(q_n^0q_n^1)^{1/2}/\sum_{j=1}^N p_j^0(q_j^0q_j^1)^{1/2} .$$

Thus the fixed basket approach to bilateral index number theory starts out with the Laspeyres and Paasche price indexes. Some form of averaging of these two indexes is called for since both indexes are equally plausible. Averaging these two indexes directly

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[17] Bowley was an early advocate of taking a symmetric average of the Paasche and Laspeyres indexes: "If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation." Arthur L. Bowley (1901; 227). Fisher (1911; 418-419) (1922) considered taking the arithmetic, geometric and harmonic averages of the Paasche and Laspeyres indexes.

[18] Walsh (1901) (1921a) and Fisher (1922) considered both averaging strategies in their classic studies on index numbers.

[19] Note that we have chosen the mean function $m(q_n^0,q_n^1)$ to be the same for each commodity n.

[20] Walsh endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance." C.M. Walsh (1921a; 103). His formula 6 is P_W defined by (13) and his 9 is the Fisher ideal defined by (9) above. His formula 8 is the formula $p^1\cdot q^1/p^0\cdot q^0 Q_W(p^0,p^1,q^0,q^1)$, which is known as the implicit Walsh price index where $Q_W(p^0,p^1,q^0,q^1)$ is the Walsh quantity index defined by (13) except the role of prices and quantities is interchanged. Thus although Walsh thought that his Walsh price index was the best functional form, his implicit Walsh price index and the "Fisher" formula were not far behind.

leads to the Fisher ideal index P_F defined by (9) as being "best" while a direct averaging of the two quantity baskets q^0 and q^1 leads to the Walsh price index P_W defined by (13) as being "best".

We turn now to another early approach to the index number problem.

3.3. Stochastic and Descriptive Statistics Approaches to Index Number Theory

The (unweighted) stochastic approach to the determination of the price index can be traced back to the work of Jevons (1865) (1884) and Edgeworth (1888) (1896) (1901) over a hundred years ago[21].

The basic idea behind the stochastic approach is that each price relative, p_n^1/p_n^0 for $n = 1,2,\ldots,N$, can be regarded as an estimate of a common inflation rate α between periods 0 and 1; i.e., Jevons and Edgeworth essentially assumed that

$$(14)\ p_n^1/p_n^0 = \alpha + \varepsilon_n \ ;\ n = 1,2,\ldots,N$$

where α is the common inflation rate and the ε_n are random variables with mean 0 and variance σ^2. The least squares estimator for α is the *Carli* (1804) *price index* P_C defined as

$$(15)\ P_C(p^0,p^1) \equiv \textstyle\sum_{n=1}^N (1/N)(p_n^1/p_n^0).$$

Unfortunately, P_C does not satisfy the time reversal test, i.e., $P_C(p^1,p^0) \neq 1/P_C(p^0,p^1)$[22].

Now assume that the logarithm of each price relative, $\ln(p_n^1/p_n^0)$, is an independent unbiased estimate of the logarithm of the inflation rate between periods 0 and 1, β say. Thus we have:

$$(16)\ \ln(p_n^1/p_n^0) = \beta + \varepsilon_n \ ;\ n = 1,2,\ldots,N$$

where $\beta \equiv \ln\alpha$ and the ε_n are independently distributed random variables with mean 0 and variance σ^2. The least squares or maximum likelihood estimator for β is the logarithm of the geometric mean of the price relatives. Hence the corresponding estimate for the common inflation rate α is the *Jevons* (1865) *price index* P_J defined as:

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[21] For additional references to the early literature, see Diewert (1993; 37-38) (1995b) and Balk (2008; 32-36).

[22] In fact Fisher (1922; 66) noted that $P_C(p^0,p^1)P_C(p^1,p^0) \geq 1$ unless the period 1 price vector p^1 is proportional to the period 0 price vector p^0; i.e., Fisher showed that the Carli index has a definite upward bias. Walsh (1901; 327) established this inequality for the case $N = 2$. Fisher urged users to abandon the use of the Carli index but his advice was generally ignored by statistical agencies until recently: "In fields other than index numbers it is often the best form of average to use. But we shall see that the simple arithmetic average produces one of the very worst of index numbers. And if this book has no other effect than to lead to the total abandonment of the simple arithmetic type of index number, it will have served a useful purpose." Irving Fisher (1922; 29-30).

(17) $P_J(p^0,p^1) \equiv \prod_{n=1}^N (p_n^1/p_n^0)^{1/N}$.

The Jevons price index P_J does satisfy the time reversal test and hence is much more satisfactory than the Carli index P_C. However, both the Jevons and Carli price indexes suffer from a fatal flaw: each price relative p_n^1/p_n^0 is regarded as being equally important and is given an equal weight in the index number formulae (15) and (17).[23] Keynes (1930; 76-81) also criticized the unweighted stochastic approach to index number theory on two other grounds: (i) price relatives are not distributed independently and (ii) there is no single inflation rate that can be applied to all parts of an economy; e.g., Keynes demonstrated empirically that wage rates, wholesale prices and final consumption prices all had different rates of inflation. In order to overcome the Keynesian criticisms of the unweighted stochastic approach to index numbers, it is necessary to:

- have a definite domain of definition for the index number and
- weight the price relatives by their economic importance.

Theil (1967; 136-137) proposed a solution to the lack of weighting in (15). He argued as follows. Suppose we draw price relatives at random in such a way that each dollar of expenditure in the base period has an equal chance of being selected. Then the probability that we will draw the nth price relative is equal to $s_n^0 \equiv p_n^0q_n^0/p^0\cdot q^0$, the period 0 expenditure share for commodity n. Then the overall mean (period 0 weighted) logarithmic price change is $\sum_{n=1}^N s_n^0\ln(p_n^1/p_n^0)$. Now repeat the above mental experiment and draw price relatives at random in such a way that each dollar of expenditure in period 1 has an equal probability of being selected. This leads to the overall mean (period 1 weighted) logarithmic price change of $\sum_{n=1}^N s_n^1\ln(p_n^1/p_n^0)$. Each of these measures of overall logarithmic price change seems equally valid so we could argue for taking a symmetric average of the two measures in order to obtain a final single measure of overall logarithmic price change. Theil (1967; 137) argued that a nice symmetric index number formula can be obtained if we make the probability of selection for the nth price relative equal to the arithmetic average of the period 0 and 1 expenditure shares for commodity n. Using these probabilities of selection, Theil's final measure of overall logarithmic price change is

(18) $\ln P_T(p^0,p^1,q^0,q^1) \equiv \sum_{n=1}^N (1/2)(s_n^0+s_n^1)\ln(p_n^1/p_n^0)$.

It is possible to give a *descriptive statistics* interpretation of the right hand side of (18). Define the nth logarithmic price ratio r_n by:

(19) $r_n \equiv \ln(p_n^1/p_n^0)$ for $n = 1,\ldots,N$.

Now define the discrete random variable, R say, as the random variable which can take on the values r_n with probabilities $\rho_n \equiv (1/2)(s_n^0+s_n^1)$ for $n = 1,\ldots,N$. Note that since each set of expenditure shares, s_n^0 and s_n^1, sums to one, the probabilities ρ_n will also sum to

[23] Walsh (1901) (1921a; 82-83), Fisher (1922; 43) and Keynes (1930; 76-77) all objected to the lack of weighting in the unweighted stochastic approach to index number theory.

one. It can be seen that the expected value of the discrete random variable R is $\ln P_T(p^0,p^1,q^0,q^1)$ as defined by the right hand side of (18). Thus the logarithm of the index P_T can be interpreted as *the expected value of the distribution of the logarithmic price ratios* in the domain of definition under consideration, where the N discrete price ratios in this domain of definition are weighted according to Theil's probability weights, ρ_n.

Taking antilogs of both sides of (18), we obtain the Theil price index; P_T.[24] This index number formula has a number of good properties. In particular, P_T satisfies the time reversal test (10) and the linear homogeneity test (12).[25]

Additional material on stochastic approaches to index number theory and references to the literature can be found in Selvanathan and Rao (1994), Diewert (1995), Wynne (1997), Clements, Izan and Selvanathan (2006) and Balk (2008; 32-36)

3.4. Test Approaches to Index Number Theory [26]

Recall equation (5) above, which set the value ratio, V^1/V^0, equal to the product of the price index, $P(p^0,p^1,q^0,q^1)$, and the quantity index, $Q(p^0,p^1,q^0,q^1)$. This is called the Product Test and we assume that it is satisfied. This equation means that as soon as the functional form for the price index P is determined, then (5) can be used to determine the functional form for the quantity index Q. However, a further advantage of assuming that the product test holds is that we can assume that the quantity index Q satisfies a "reasonable" property and then use (5) to translate this test on the quantity index into a corresponding test on the price index P.[27]

If N = 1, so that there is only one price and quantity to be aggregated, then a natural candidate for P is p_1^1/p_1^0 , the single price ratio, and a natural candidate for Q is q_1^1/q_1^0 , the single quantity ratio. When the number of commodities or items to be aggregated is greater than 1, then what index number theorists have done over the years is propose properties or tests that the price index P should satisfy. These properties are generally multi-dimensional analogues to the one good price index formula, p_1^1/p_1^0. Below, we list twenty-one tests that turn out to characterize the Fisher ideal price index.

We shall assume that every component of each price and quantity vector is positive; i.e., $p^t >> 0_N$ and $q^t >> 0_N$ for t = 0,1. If we want to set $q^0 = q^1$, we call the common quantity vector q; if we want to set $p^0 = p^1$, we call the common price vector p.

Our first two tests are not very controversial and so we will not discuss them.

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[24] This index first appeared explicitly as formula 123 in Fisher (1922; 473). P_T is generally attributed to Törnqvist (1936) but this article did not have an explicit definition for P_T; it was defined explicitly in Törnqvist and Törnqvist (1937); see Balk (2008; 26).

[25] For a listing of some of the tests that P_T, P_F, and P_W satisfy, see Diewert (1992; 223). In Fisher (1922), these indexes were listed as numbers 123, 353 and 1153 respectively.

[26] The material in this section is based on Diewert (1992) where more detailed references to the literature on the origins of the various tests can be found.

[27] This observation was first made by Fisher (1911; 400-406). Vogt (1980) also pursued this idea.

T1: *Positivity*: $P(p^0,p^1,q^0,q^1) > 0$.

T2: *Continuity*: $P(p^0,p^1,q^0,q^1)$ is a continuous function of its arguments.

Our next two tests are somewhat more controversial.

T3: *Identity or Constant Prices Test*: $P(p,p,q^0,q^1) = 1$.

That is, if the price of every good is identical during the two periods, then the price index should equal unity, no matter what the quantity vectors are. The controversial part of this test is that the two quantity vectors are allowed to be different in the above test.[28]

T4: *Fixed Basket or Constant Quantities Test*: $P(p^0,p^1,q,q) = \sum_{i=1}^N p_i^1 q_i / \sum_{i=1}^N p_i^0 q_i$.

That is, if quantities are constant during the two periods so that $q^0 = q^1 \equiv q$, then the price index should equal the expenditure on the constant basket in period 1, $\sum_{i=1}^N p_i^1 q_i$, divided by the expenditure on the basket in period 0, $\sum_{i=1}^N p_i^0 q_i$.

The following four tests are *homogeneity tests* and they restrict the behavior of the price index P as the scale of any one of the four vectors p^0,p^1,q^0,q^1 changes.

T5: *Proportionality in Current Prices*: $P(p^0,\lambda p^1,q^0,q^1) = \lambda P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 1 prices are multiplied by the positive number λ, then the new price index is λ times the old price index. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree one in the components of the period 1 price vector p^1. Most index number theorists regard this property as a very fundamental one that the index number formula should satisfy.

Walsh (1901) and Fisher (1911; 418) (1922; 420) proposed the related proportionality test $P(p,\lambda p,q^0,q^1) = \lambda$. This last test is a combination of T3 and T5; in fact Walsh (1901, 385) noted that this last test implies the identity test, T3.

In our next test, instead of multiplying all period 1 prices by the same number, we multiply all period 0 prices by the number λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1,q^0,q^1) = \lambda^{-1} P(p^0,p^1,q^0,q^1)$ for $\lambda > 0$.

That is, if all period 0 prices are multiplied by the positive number λ, then the new price index is $1/\lambda$ times the old price index. Put another way, the price index function

[28] Usually, economists assume that given a price vector p, the corresponding quantity vector q is uniquely determined. Here, we have the same price vector but the corresponding quantity vectors are allowed to be different.

$P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree minus one in the components of the period 0 price vector p^0.

The following two homogeneity tests can also be regarded as invariance tests.

T7: *Invariance to Proportional Changes in Current Quantities*: $P(p^0,p^1,q^0,\lambda q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if current period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 1 quantity vector q^1. Vogt (1980, 70) was the first to propose this test and his derivation of the test is of some interest. Suppose the quantity index Q satisfies the quantity analogue to the price test T5; i.e., suppose Q satisfies $Q(p^0,p^1,q^0,\lambda q^1) = \lambda Q(p^0,p^1,q^0,q^1)$ for $\lambda > 0$. Then using the product test (5), we see that P must satisfy T7.

T8: *Invariance to Proportional Changes in Base Quantities*: $P(p^0,p^1,\lambda q^0,q^1) = P(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$.

That is, if base period quantities are all multiplied by the number λ, then the price index remains unchanged. Put another way, the price index function $P(p^0,p^1,q^0,q^1)$ is (positively) homogeneous of degree zero in the components of the period 0 quantity vector q^0. If the quantity index Q satisfies the following counterpart to T8: $Q(p^0,p^1,\lambda q^0,q^1) = \lambda^{-1}Q(p^0,p^1,q^0,q^1)$ for all $\lambda > 0$, then using (5), the corresponding price index P must satisfy T8. This argument provides some additional justification for assuming the validity of T8 for the price index function P.

T7 and T8 together impose the property that the price index P does not depend on the *absolute* magnitudes of the quantity vectors q^0 and q^1.

The next five tests are *invariance* or *symmetry tests*. Fisher (1922; 62-63, 458-460) and Walsh (1921b; 542) seem to have been the first researchers to appreciate the significance of these kinds of tests. Fisher (1922, 62-63) spoke of fairness but it is clear that he had symmetry properties in mind. It is perhaps unfortunate that he did not realize that there were more symmetry and invariance properties than the ones he proposed; if he had realized this, it is likely that he would have been able to provide an axiomatic characterization for his ideal price index. Our first invariance test is that the price index should remain unchanged if the *ordering* of the commodities is changed:

T9: *Commodity Reversal Test* (or invariance to changes in the ordering of commodities):

$$P(p^{0*},p^{1*},q^{0*},q^{1*}) = P(p^0,p^1,q^0,q^1)$$

where p^{t*} denotes a permutation of the components of the vector p^t and q^{t*} denotes the same permutation of the components of q^t for $t = 0,1$. This test is due to Irving Fisher (1922), and it is one of his three famous reversal tests. The other two are the time reversal test and the factor reversal test which will be considered below.

T10: *Invariance to Changes in the Units of Measurement* (commensurability test):

$$P(\alpha_1 p_1^0,...,\alpha_N p_N^0; \alpha_1 p_1^1,...,\alpha_N p_N^1; \alpha_1^{-1} q_1^0,...,\alpha_N^{-1} q_N^0; \alpha_1^{-1} q_1^1,...,\alpha_N^{-1} q_N^1) =$$
$$P(p_1^0,...,p_N^0; p_1^1,...,p_N^1; q_1^0,...,q_N^0; q_1^1,...,q_N^1) \text{ for all } \alpha_1 > 0, \ldots, \alpha_N > 0.$$

That is, the price index does not change if the units of measurement for each commodity are changed. The concept of this test was due to Jevons (1884; 23) and the Dutch economist Pierson (1896; 131), who criticized several index number formula for not satisfying this fundamental test. Fisher (1911; 411) first called this test *the change of units test* and later, Fisher (1922; 420) called it the *commensurability test*.

T11: *Time Reversal Test*: $P(p^0,p^1,q^0,q^1) = 1/P(p^1,p^0,q^1,q^0)$.

That is, if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index. Obviously, in the one good case when the price index is simply the single price ratio; this test is satisfied (as are all of the other tests listed in this section). When the number of goods is greater than one, many commonly used price indexes fail this test; e.g., the Laspeyres (1871) price index, P_L defined earlier by (7), and the Paasche (1874) price index, P_P defined earlier by (8), both *fail* this fundamental test. The concept of the test was due to Pierson (1896; 128), who was so upset with the fact that many of the commonly used index number formulae did not satisfy this test, that he proposed that the entire concept of an index number should be abandoned. More formal statements of the test were made by Walsh (1901; 368) (1921b; 541) and Fisher (1911; 534) (1922; 64).

Our next two tests are more controversial, since they are not necessarily consistent with the economic approach to index number theory. However, these tests are quite consistent with the weighted stochastic approach to index number theory discussed earlier in section 3.3.

T12: *Quantity Reversal Test* (quantity weights symmetry test):

$$P(p^0,p^1,q^0,q^1) = P(p^0,p^1,q^1,q^0).$$

That is, if the quantity vectors for the two periods are interchanged, then the price index remains invariant. This property means that if quantities are used to weight the prices in the index number formula, then the period 0 quantities q^0 and the period 1 quantities q^1 must enter the formula in a symmetric or even handed manner. Funke and Voeller (1978; 3) introduced this test; they called it the *weight property*.

The next test is the analogue to T12 applied to quantity indexes:

T13: *Price Reversal Test* (price weights symmetry test):

$$\{\sum_{i=1}^N p_i^1 q_i^1 / \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) = \{\sum_{i=1}^N p_i^0 q_i^1 / \sum_{i=1}^N p_i^1 q_i^0\}/P(p^1,p^0,q^0,q^1).$$

Thus if we use (5) to define the quantity index Q in terms of the price index P, then it can be seen that T13 is equivalent to the following property for the associated quantity index Q: $Q(p^0,p^1,q^0,q^1) = Q(p^1,p^0,q^0,q^1)$. That is, if the price vectors for the two periods are interchanged, then the quantity index remains invariant. Thus if prices for the same good in the two periods are used to weight quantities in the construction of the quantity index, then property T13 implies that these prices enter the quantity index in a symmetric manner.

The next three tests are mean value tests.

T14: *Mean Value Test for Prices*:

$$\min_i (p_i^1/p_i^0 : i=1,...,N) \leq P(p^0,p^1,q^0,q^1) \leq \max_i (p_i^1/p_i^0 : i = 1,...,N).$$

That is, the price index lies between the minimum price ratio and the maximum price ratio. Since the price index is supposed to be some sort of an average of the N price ratios, p_i^1/p_i^0, it seems essential that the price index P satisfy this test.

The next test is the analogue to T14 applied to quantity indexes:

T15: *Mean Value Test for Quantities*:

$$\min_i (q_i^1/q_i^0 : i=1,...,n) \leq \{V^1/V^0\}/ P(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,n)$$

where V^t is the period t value aggregate $V^t \equiv \sum_{n=1}^{N} p_n^t q_n^t$ for t = 0,1. Using (5) to define the quantity index Q in terms of the price index P, we see that T15 is equivalent to the following property for the associated quantity index Q:

$$(20)\ \min_i (q_i^1/q_i^0 : i=1,...,N) \leq Q(p^0,p^1,q^0,q^1) \leq \max_i (q_i^1/q_i^0 : i = 1,...,N).$$

That is, the implicit quantity index Q defined by P lies between the minimum and maximum rates of growth q_i^1/q_i^0 of the individual quantities.

In section 3.2, we argued that it was very reasonable to take an average of the Laspeyres and Paasche price indexes as a single "best" measure of overall price change. This point of view can be turned into a test:

T16: *Paasche and Laspeyres Bounding Test*: The price index P lies between the Laspeyres and Paasche indices, P_L and P_P, defined earlier by (7) and (8) above.

The final four tests are monotonicity tests; i.e., how should the price index $P(p^0,p^1,q^0,q^1)$ change as any component of the two price vectors p^0 and p^1 increases or as any component of the two quantity vectors q^0 and q^1 increases.

T17: *Monotonicity in Current Prices*: $P(p^0,p^1,q^0,q^1) < P(p^0,p^2,q^0,q^1)$ if $p^1 < p^2$.

That is, if some period 1 price increases, then the price index must increase, so that $P(p^0,p^1,q^0,q^1)$ is increasing in the components of p^1. This property was proposed by

Eichhorn and Voeller (1976; 23) and it is a very reasonable property for a price index to satisfy.

T18: *Monotonicity in Base Prices*: $P(p^0,p^1,q^0,q^1) > P(p^2,p^1,q^0,q^1)$ if $p^0 < p^2$.

That is, if any period 0 price increases, then the price index must decrease, so that $P(p^0,p^1,q^0,q^1)$ is decreasing in the components of p^0 . This very reasonable property was also proposed by Eichhorn and Voeller (1976; 23).

T19: *Monotonicity in Current Quantities*: if $q^1 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) < \{\sum_{i=1}^N p_i^1 q_i^2/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^2)$.

T20: *Monotonicity in Base Quantities*: if $q^0 < q^2$, then
$\{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0\}/P(p^0,p^1,q^0,q^1) > \{\sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^2\}/P(p^0,p^1,q^2,q^1)$.

If we define the implicit quantity index Q that corresponds to P using (5), we find that T19 translates into the following inequality involving Q:

(21) $Q(p^0,p^1,q^0,q^1) < Q(p^0,p^1,q^0,q^2)$ if $q^1 < q^2$.

That is, if any period 1 quantity increases, then the implicit quantity index Q that corresponds to the price index P must increase. Similarly, we find that T20 translates into:

(22) $Q(p^0,p^1,q^0,q^1) > Q(p^0,p^1,q^2,q^1)$ if $q^0 < q^2$.

That is, if any period 0 quantity increases, then the implicit quantity index Q must decrease. Tests T19 and T20 are due to Vogt (1980, 70).

The final test is Irving Fisher's (1921; 534) (1922; 72-81) third reversal test (the other two being T9 and T11):

T21: *Factor Reversal Test* (functional form symmetry test):
$P(p^0,p^1,q^0,q^1)\, P(q^0,q^1,p^0,p^1) = \sum_{i=1}^N p_i^1 q_i^1/ \sum_{i=1}^N p_i^0 q_i^0 = V^1/V^0$.

A justification for this test is the following one: if $P(p^0,p^1,q^0,q^1)$ is a good functional form for the price index, then if we reverse the roles of prices and quantities, $P(q^0,q^1,p^0,p^1)$ ought to be a good functional form for a quantity index (which seems to be a correct argument) and thus the product of the price index $P(p^0,p^1,q^0,q^1)$ and the quantity index $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ ought to equal the value ratio, V^1/V^0 . The second part of this argument does not seem to be valid and thus many researchers over the years have objected to the factor reversal test.

It is straightforward to show that the Fisher ideal price index P_F defined earlier by (9) satisfies all 21 tests. Is this the only index number formula that satisfies all of these tests? The answer is yes: Funke and Voeller (1978; 180) showed that the only index number function $P(p^0,p^1,q^0,q^1)$ which satisfies T1 (positivity), T11 (time reversal test), T12 (quantity reversal test) and T21 (factor reversal test) is the Fisher ideal index P_F defined by (9). Diewert (1992; 221) proved a similar result: namely that if P satisfied T1 and the three reversal tests T11-T13, then P must equal P_F.

Thus it seems that from the perspective of the above test approach to index number theory, the Fisher ideal index satisfies more "reasonable" tests than competing indexes and hence can be regarded as "best" from the viewpoint of this perspective.

There is another perspective to the test approach to index number theory. The above approach looked at axioms or tests that pertained to situations where the price index was a function of the two price vectors, p^0 and p^1, and the two matching quantity vectors, q^0 and q^1. In this framework, the two quantity vectors essentially act as weights for the prices. However, there is an alternative framework where the price index, say $P^*(p^0,p^1,e^0,e^1)$, is regarded as a function of the two price vectors, p^0 and p^1, and the two matching *expenditure vectors*, e^0 and e^1.[29] An axiomatic approach to the determination of the functional form for indexes of this type is developed in the ILO (2004; 307-309) and the Törnqvist index defined earlier by (18) emerges as "best" from the perspective of this second test approach to index number theory. Thus both the Fisher and Törnqvist indexes can be given strong axiomatic justifications.

There is one final important test that should be added to the above list of tests and that is the following *circularity test* [30] which involves looking at the prices and quantities that pertain to three periods:

T22: *Circularity Test*: $P(p^0,p^1,q^0,q^1)\, P(p^1,p^2,q^1,q^2) = P(p^0,p^2,q^0,q^2)$.

If this test is satisfied, then the rate of price change going from period 0 to 1, $P(p^0,p^1,q^0,q^1)$, times the rate of price change going from period 1 to 2, $P(p^1,p^2,q^1,q^2)$, is equal to the rate of price change going from period 0 to 2 directly, $P(p^0,p^2,q^0,q^2)$. If there is only one commodity in the aggregate, then the price index $P(p^0,p^1,q^0,q^1)$ just becomes the single price ratio, p_1^1/p_1^0, and the circularity test T22 becomes the equation $[p_1^1/p_1^0][p_1^2/p_1^1] = [p_1^2/p_1^0]$, which is obviously satisfied. The equation in the circularity test illustrates the difference between chained index numbers and fixed base index numbers. The left hand side of T22 uses the *chain principle* to construct the overall inflation between periods 0 and 2 whereas the right hand side uses the *fixed base principle* to construct an estimate of the overall price change between periods 0 and 1.[31]

[29] Component n of the period t expenditure vector e^t is defined as $e_n^t \equiv p_n^t q_n^t$ for n = 1,...,N and t = 0,1. Thus if the price components p_n^t are known, then a knowledge of either the quantity components q_n^t or the expenditure components e_n^t will determine prices, quantities and expenditures in both periods.

[30] The test name is due to Fisher (1922; 413) and the concept was originally due to Westergaard (1890; 218-219).

[31] Thus when the chain principle is used, the price index $P(p^t,p^{t+1},q^t,q^{t+1})$ is used to update the period t index level to construct the period t+1 index level, whereas the fixed base system constructs the period t+1 index level relative to period 0 directly as $P(p^0,p^{t+1},q^0,q^{t+1})$, where the period 0 level is set equal to 1. Fisher (1911; 203) introduced this fixed base and chain terminology. The concept of chaining is due to Lehr (1885) and Marshall (1887; 373).

It would be good if our preferred index number formulae, the Fisher, Walsh and Törnqvist indexes (P_F, P_W and P_T), satisfied the circularity test but unfortunately, none of these indexes satisfy T22. Thus if any of these indexes are used by a statistical agency, then the question arises: should the sequence of index values be computed using fixed base indexes or chained indexes? The remainder of this section will attempt to address this question.

The first point to note is that fixed base indexes cannot be used for long periods of time in today's dynamic economy where new commodities appear and older ones become obsolete. Under these conditions, it becomes increasingly difficult to match commodity prices over long periods of time and index number theory is dependent on a high degree of matching of the prices between the two periods being compared. However, this possible lack of matching does not rule out using fixed base indexes for shorter periods of time, say over a year or two.

The main advantage of using chained indexes is that if prices and quantities are trending relatively smoothly, chaining will reduce the spread between the Paasche and Laspeyres indexes.[32] These two indexes each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth". Since annual data generally has smooth trends, the use of chained indexes is generally appropriate at this level of aggregation; see Hill (1993; 136-137).

However, the story is different at subannual levels; i.e., if the index is to be produced at monthly or quarterly frequencies. Hill (1993; 388), drawing on the earlier research of Szulc (1983) and Hill (1988; 136-137), noted that it is not appropriate to use the chain system when prices oscillate or "bounce" to use Szulc's (1983; 548) term. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of sales. The *price bouncing problem* or the problem of *chain drift* can be illustrated if we make use of the following test due to Walsh (1901; 389), (1921b; 540) (1924; 506):[33]

T23: *Multiperiod Identity Test*: $P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2)P(p^2,p^0,q^2,q^0) = 1$.

Thus price change is calculated over consecutive periods but an artificial final period is introduced where the prices and quantities revert back to the prices and quantities in the very first period. The Walsh test T23 asks that the product of all of these price changes should equal unity. If prices have no definite trends but are simply bouncing up and down in a range, then the above test can be used to evaluate the amount of chain drift that

[32] See Diewert (1978; 895) and Hill (1988) (1993; 387-388). Chaining under these conditions will also reduce the spread between fixed base and chained indexes using P_F, P_W or P_T as the basic bilateral formula.
[33] This is Diewert's (1993; 40) term for the test. Walsh did not limit himself to just three periods as in T23; he considered an indefinite number of periods. If tests T3 and T22 are satisfied, then T23 will also be satisfied.

occurs if chained indexes are used under these conditions. *Chain drift* occurs when an index does not return to unity when prices in the current period return to their levels in the base period; see the ILO (2004; 445). Fixed base indexes operating under these conditions will not be subject to chain drift.

It is possible to be a bit more precise under what conditions one should chain or not chain. Basically, one should chain if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indexes at each link. Of course, one needs a measure of how similar are the prices and quantities pertaining to two periods. A practical problem with this *similarity linking approach* is: exactly how should the measure of price or quantity similarity be measured?[34] For *annual* time series data, it turns out that for various "reasonable" similarity measures, chained indexes are generally consistent with the similarity approach to linking observations. However, for subannual data, it is generally better to use fixed base indexes in order to eliminate the problem of chain drift.

We conclude this subsection with a discussion on how well our best indexes, P_F, P_W and P_T defined by (9), (13) and (18) above, satisfy the circularity test, T22. Fisher (1922; 277) found that for his annual data set, the Fisher ideal index P_F satisfied circularity to a reasonably high degree of approximation. It turns out that this result generally holds using annual data for P_W and P_T as well. It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for these three indexes. Alterman, Diewert and Feenstra (1999; 61) showed that if the logarithmic price ratios ln (p_n^t/p_n^{t-1}) trend linearly with time t and the expenditure shares s_i^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test *exactly*.[35] Since many economic time series on prices and quantities satisfy these assumptions approximately, the above exactness result will imply that the Törnqvist index P_T will satisfy the circularity test approximately. But Diewert (1978; 888) showed that P_T, P_F and P_W numerically approximate each other to the second order around an equal price and quantity point and so these three indexes will generally be very close to each other using annual time series data. Hence since P_T will generally satisfy the circularity test to some degree of approximation, P_F and P_W will also satisfy circularity approximately in the time series context using annual data. Thus for *annual* economic time series, P_F, P_T and P_W will generally satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle. However, this same conclusion does *not* hold for *subannual* data that has substantial period to period fluctuations in prices. For fluctuating subannual data, chained indexes can give very unsatisfactory results; i.e., Walsh's multiperiod identity test will be far from being

[34] This similarity approach to linking bilateral comparisons into a complete set of comparisons across all observations has been pioneered by Robert Hill (1999a) (1999b) (2001) (2004) (2009). For an axiomatic approach to similarity measures, see Diewert (2009b).

[35] This exactness result can be extended to cover the case when there are monthly proportional variations in prices and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999; 65).

satisfied. Under these conditions, fixed base indexes or multilateral methods should be used.[36]

3.5. The Economic Approach to Consumer Price Indexes

In this subsection, we will outline the theory of the cost of living index for a single consumer (or household) that was first developed by the Russian economist, A. A. Konüs (1924).[37] This theory relies on the assumption of *optimizing behavior* on the part of the consumer. Thus given a vector of commodity or input prices p^t that the consumer faces in a given time period t, it is assumed that the corresponding observed quantity vector q^t is the solution to a cost minimization problem that involves the consumer's preference or utility function f.

The economic approach assumes that "the" consumer has well defined *preferences* over different combinations of the N consumer commodities or items. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a nonnegative, continuous, increasing, and quasiconcave utility function f, which is defined over the nonnegative orthant. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods t = 0,1. Thus the observed period t consumption vector q^t solves the following *period t cost minimization problem*:

$$(23)\quad C(u^t,p^t) \equiv \min_q \{p^t \cdot q : f(q) = u^t\} = p^t \cdot q^t ; \qquad t = 0,1.$$

The period t price vector for the N commodities under consideration that the consumer faces is p^t. The Konüs (1939) family of *true cost of living indexes* $P_K(p^0,p^1,q)$ between periods 0 and 1 is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv f(q)$ where q is a positive reference quantity vector:

$$(24)\quad P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0].$$

We say that definition (24) defines a *family* of price indexes because there is one such index for each reference quantity vector q chosen. However, if we place an additional restriction on the utility function f, then it turns out that the Konüs price index, $P_K(p^0,p^1,q)$, will no longer depend on the reference q.

The extra assumption on f is that f be (positively) *linearly homogeneous* so that $f(\lambda q) = \lambda f(q)$ for all $\lambda > 0$ and all $q \geq 0_N$. In the economics literature, this extra assumption is known as the assumption of *homothetic preferences*.[38] Under this assumption, the consumer's cost function, C(u,p) decomposes into uc(p) where c(p) is the consumer's

[36] See Szulc (1983), Hill (1988) and Ivancic, Diewert and Fox (2011).
[37] For extensions to the case of many households, see Diewert (2001).
[38] More precisely, Shephard (1953) defined a homothetic function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence, we simply identify the homothetic preferences assumption with the linear homogeneity assumption.

unit cost function, $c(p) \equiv C(1,p)$, which corresponds to f. Under the assumption of cost minimizing behavior in both periods, it can be shown that the homotheticity assumption implies that equations (23) simplify to the following equations:

(25) $p^t \cdot q^t = c(p^t)f(q^t)$ for t = 0,1.

Thus under the linear homogeneity assumption on the utility function f, observed period t expenditure on the n commodities is equal to the period t unit cost $c(p^t)$ of achieving one unit of utility times the period t utility level, $f(q^t)$. Obviously, we can identify the period t unit cost, $c(p^t)$, as the period t price level P^t and the period t level of utility, $f(q^t)$, as the period t quantity level Q^t (as in section 3.1 above).

The linear homogeneity assumption on the consumer's preference function f leads to a simplification for the family of Konüs true cost of living indexes, $P_K(p^0,p^1,q)$, defined by (24) above. Using this definition for an arbitrary reference quantity vector q and the decomposition $C(f(q),p^t) = c(p^t)f(q)$ for t = 0,1, we have:

(26) $P_K(p^0,p^1,q) \equiv C[f(q),p^1]/C[f(q),p^0] = c(p^1)f(q)/c(p^0)f(q) = c(p^1)/c(p^0)$.

Thus under the homothetic preferences assumption, the entire family of Konüs true cost of living indexes collapses to a single index, $c(p^1)/c(p^0)$, which is the ratio of the minimum costs of achieving a unit utility level when the consumer faces period 1 and 0 prices respectively.

If we use the Konüs true cost of living index defined by the right hand side of (26) as our price index concept, then the corresponding implicit quantity index can be defined as the value ratio divided by the Konüs price index:

(27) $Q(p^0,p^1,q^0,q^1,q) \equiv p^1 \cdot q^1/[p^0 \cdot q^0\ P_K(p^0,p^1,q)] = f(q^1)/f(q^0)$.

Thus under the homothetic preferences assumption, the *implicit quantity index* that corresponds to the true cost of living price index $c(p^1)/c(p^0)$ is the *utility ratio* $f(q^1)/f(q^0)$. Since the utility function is assumed to be homogeneous of degree one, this is the natural definition for a quantity index.[39]

Recall that the Fisher price index, $P_F(p^0,p^1,q^0,q^1)$, was defined by (9). The companion *Fisher quantity index*, $Q_F(p^0,p^1,q^0,q^1)$, can be defined using (5). Now suppose that the consumer's preferences can be represented by the homothetic utility function f defined as

(28) $f(q) \equiv [q^TAq]^{1/2}$

where $A \equiv [a_{ij}]$ is an N by N symmetric matrix that has one positive eigenvalue (that has a strictly positive eigenvector) and the remaining N−1 eigenvalues are zero or negative. Under these conditions, there will be a *region of regularity* where the function f is

[39] Samuelson and Swamy (1974) used this homothetic approach to index number theory.

positive, concave and increasing and hence f can provide a valid representation of preferences over this region. Using these preferences and the assumption of cost minimizing behavior in periods 0 and 1, it can be shown that

(29) $Q_F(p^0,p^1,q^0,q^1) = f(q^1)/f(q^0)$.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function f defined by (28), the Fisher ideal quantity index Q_F is *exactly* equal to the true quantity index, $f(q^1)/f(q^0)$.[40]

Let c(p) be the unit cost function that corresponds to the homogeneous quadratic utility function f defined by (28). Then using (5), (9), (25) and (29), it can be shown that

(30) $P_F(p^0,p^1,q^0,q^1) = c(p^1)/c(p^0)$.

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function $f(q) = (q^TAq)^{1/2}$, the Fisher ideal price index P_F is *exactly* equal to the true price index, $c(p^1)/c(p^0)$. The significance of (29) and (30) is that we can calculate the consumer's true rate of utility growth and his or her true rate of price inflation *without having to undertake any econometric estimation*; i.e., the left hand sides of (29) and (30) can be calculated exactly using observable price and quantity data for the consumer for the two periods under consideration. Thus the present economic approach to index number theory using a *ratio approach* leads to practical solutions to the index number problem whereas the earlier *levels approach* explained in section 3.1 did not lead to practical solutions.

A twice continuously differentiable function f(q) of N variables q can provide a *second order approximation* to another such function $f^*(q)$ around the point q^* if the level and all of the first and second order partial derivatives of the two functions coincide at q^*. It can be shown[41] that the homogeneous quadratic function f can provide a second order approximation to an arbitrary f^* around any point q^* in the class of twice continuously differentiable linearly homogeneous functions. Thus the homogeneous quadratic functional form defined by (28) is a *flexible functional form.*[42] Diewert (1976; 117) termed an index number formula $Q_F(p^0,p^1,q^0,q^1)$ that was *exactly* equal to the true quantity index $f(q^1)/f(q^0)$ (where f is a flexible functional form) *a superlative index number formula*.[43] Equation (29), and the fact that the homogeneous quadratic function f defined by (28) is a flexible functional form, shows that the Fisher ideal quantity index Q_F is a superlative index number formula. Since the Fisher ideal price index P_F also

[40] This result was first derived by Konüs and Byushgens (1926). For an alternative derivation and the early history of this result, see Diewert (1976; 116).

[41] See Diewert (1976; 130) and let the parameter r equal 2.

[42] Diewert (1974; 133) introduced this term to the economics literature.

[43] As we have seen earlier, Fisher (1922; 247) used the term superlative to describe the Fisher ideal price index. Thus Diewert adopted Fisher's terminology but attempted to give more precision to Fisher's definition of superlativeness.

satisfies (30) where c(p) is the dual unit cost function that is generated by the homogeneous quadratic utility function, P_F is also a superlative index number formula.

It turns out that there are many other superlative index number formulae; i.e., there exist many quantity indexes $Q(p^0,p^1,q^0,q^1)$ that are exactly equal to $f(q^1)/f(q^0)$ and many price indexes $P(p^0,p^1,q^0,q^1)$ that are exactly equal to $c(p^1)/c(p^0)$ where the aggregator function f or the unit cost function c is a flexible functional form. We will define a family of superlative indexes below.

Suppose that the consumer has the *following quadratic mean of order r utility function*:[44]

$$(31)\quad f^r(q_1,\ldots,q_N) \equiv [\sum_{i=1}^N\sum_{k=1}^N a_{ik}\, q_i^{r/2}\, q_k^{r/2}\,]^{1/r}$$

where the parameters a_{ik} satisfy the symmetry conditions $a_{ik} = a_{ki}$ for all i and k and the parameter r satisfies the restriction $r \neq 0$. Diewert (1976; 130) showed that the utility function f^r defined by (49) is a flexible functional form; i.e., it can approximate an arbitrary twice continuously differentiable linearly homogeneous functional form to the second order.[45] Note that when $r = 2$, f^r equals the homogeneous quadratic function defined by (28) above.

Define the *quadratic mean of order r quantity index* Q^r by:

$$(32)\quad Q^r(p^0,p^1,q^0,q^1) \equiv \{\sum_{i=1}^N s_i^0\,(q_i^1/q_i^0)^{r/2}\}^{1/r}\,\{\sum_{i=1}^N s_i^1\,(q_i^1/q_i^0)^{-r/2}\}^{-1/r}$$

where $s_i^t \equiv p_i^tq_i^t/\sum_{k=1}^N p_k^tq_k^t$ is the period t expenditure share for commodity i. It can be verified that when $r = 2$, Q^r simplifies to Q_F, the Fisher ideal quantity index. It can be shown that Q^r is exact for the aggregator function f^r defined by (31); i.e., we have

$$(33)\quad Q^r(p^0,p^1,q^0,q^1) = f^r(q^1)/f^r(q^0).$$

Thus under the assumption that the consumer engages in cost minimizing behavior during periods 0 and 1 and has preferences over the N commodities that correspond to the utility function defined by (31), the quadratic mean of order r quantity index Q_F is *exactly* equal to the true quantity index, $f^r(q^1)/f^r(q^0)$.[46] Since Q^r is exact for f^r and f^r is a flexible functional form, we see that the quadratic mean of order r quantity index Q^r is a *superlative index* for each $r \neq 0$. Thus there are an infinite number of superlative quantity indexes.

For each quantity index Q^r, we can use (5) in order to define the corresponding *implicit quadratic mean of order r price index* P^r:

[44] This terminology is due to Diewert (1976; 129).

[45] This result holds for any predetermined $r \neq 0$; i.e., we require only the $N(N+1)/2$ independent a_{ik} parameters in order to establish the flexibility of f^r in the class of linearly homogeneous aggregator functions.

[46] See Diewert (1976; 130).

(34) $P^r(p^0,p^1,q^0,q^1) \equiv p^1 \cdot q^1/[p^0 \cdot q^0 Q^r(p^0,p^1,q^0,q^1)] = c^r(p^1)/c^r(p^0)$

where c^r is the unit cost function that is dual to the aggregator function f^r defined by (31) above. For each $r \neq 0$, the implicit quadratic mean of order r price index P^r is also a superlative index.

When r = 2, Q^r defined by (50) simplifies to Q_F, the Fisher ideal quantity index and P^r defined by (52) simplifies to P_F, the Fisher ideal price index. When r = 1, Q^r defined by (32) simplifies to

(35) $Q^1(p^0,p^1,q^0,q^1) = [p^1 \cdot q^1/p^0 \cdot q^0]/P_W(p^0,p^1,q^0,q^1)$

where P_W is the *Walsh* (1901; 398) (1921a; 97) *price index* defined earlier by (13). Thus the Walsh price index is also a superlative price index.

The above results provide reasonably strong justifications for the Fisher and Walsh price indexes from the viewpoint of the economic approach. An even stronger justification[47] can be provided for the Törnqvist Theil index P_T defined by (18) as we will show below.

Suppose that the consumer's cost function, C(u,p), has the following *translog functional form*:[48]

$$(36)\quad \ln C(u,p) \equiv a_0 + \sum_{i=1}^N a_i \ln p_i + (1/2) \sum_{i=1}^N \sum_{k=1}^N a_{ik} \ln p_i \ln p_k + b_0 \ln u + \sum_{i=1}^N b_i \ln p_i \ln u + (1/2)\, b_{00} [\ln u]^2$$

where ln is the natural logarithm function and the parameters a_i, a_{ik}, and b_i satisfy the following restrictions: (i) $a_{ik} = a_{ki}$ for i,k = 1,…,N; (ii) $\sum_{i=1}^N a_i = 1$; (iii) $\sum_{i=1}^N b_i = 0$; (iv) $\sum_{k=1}^N a_{ik} = 0$ for i = 1,…,N. These restrictions ensure that C(u,p) defined by (36) is linearly homogeneous in p. It can be shown that this translog cost function can provide a second order Taylor series approximation to an arbitrary cost function.[49]

We assume that the consumer engages in cost minimizing behavior during periods 0 and 1 and has the preferences that are dual to the translog cost function defined by (36). Define the geometric average of the period 0 and 1 utility levels as $u^* \equiv [u^0u^1]^{1/2}$. Then it can be shown that the log of P_T defined by (19) is exactly equal to the log of the Konüs true cost of living index that corresponds to the reference indifference surface that is indexed by the intermediate utility level u^*; i.e., we have the following exact identity:[50]

[47] The exact index number formula (55) is stronger than the above results because we no longer have to assume homothetic preferences.
[48] Christensen, Jorgenson and Lau (1975) and Diewert (1976) introduced this function into the economics and index number literature.
[49] It can also be shown that if $b_0 = 1$ and all of the $b_i = 0$ for i = 1,...,N and $b_{00} = 0$, then $C(u,p) = uC(1,p) \equiv uc(p)$; i.e., with these additional restrictions on the parameters of the general translog cost function, we have homothetic preferences.
[50] This result is due to Diewert (1976; 122).

(37) $C(u^*,p^1)/C(u^*,p^0) = P_T(p^0,p^1,q^0,q^1)$.

Since the translog cost function is a flexible functional form, the Törnqvist-Theil price index P_T is also a *superlative index*.[51] The importance of (37) as compared to the earlier exact index number results is that it is no longer necessary to assume that preferences are homothetic. However, it is necessary to choose the reference utility level on the left hand side of (37) to be the geometric mean of u^0 and u^1 in order to obtain the new exact index number result.[52]

It is somewhat mysterious how a ratio of *unobservable* cost functions of the form appearing on the left hand side of the above equation can be *exactly* estimated by an *observable* index number formula but the key to this mystery is the assumption of cost minimizing behavior and the quadratic nature of the underlying preferences. In fact, all of the exact index number results derived in this section can be derived using transformations of a quadratic identity.[53]

The important message to take home from this subsection is that the Fisher, Walsh and Theil indexes, P_F, P_W and P_T, can all be given strong justifications from the viewpoint of the economic approach to index number theory. Note that these same formulae also emerged as being "best" from the viewpoints of the basket, stochastic and test approaches to index number theory. *Thus the four major approaches to bilateral index number theory lead to the same three formulae as being best*. Which formula should then be used by a statistical agency as their target index? It turns out that for "typical" time series data, it will not matter much, since the three indexes will generally numerically approximate each other very closely.[54]

We now turn our attention to test and stochastic approaches to the determination of "best" functional forms for elementary indexes.

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[51] Diewert (1978; 888) showed that $P_T(p^0,p^1,q^0,q^1)$ approximates the other superlative indexes P^r and P^{r*} to the second order around an equal price and quantity point.

[52] For exact index number results in the context of quantity indexes and nonhomothetic preferences that are analogous to (37), see Diewert (1976; 123-124) and Diewert (2009a; 241) where the first paper uses Malmquist (1953) quantity indexes and the second one uses Allen (1949) quantity indexes. It is also possible to generalize the result (37) to situations where the consumer changes his or her tastes going from period 0 to period 1. Again, under the assumption that the consumer has (possibly different) translog preferences in each period, it can be shown that the Törnqvist price index P_T is exactly equal to the geometric mean of two separate price indexes where the tastes for one period are used in one true cost of living index and the tastes for the other period are used in the other true cost of living index. There are some restrictions on the degree of difference in the preferences over the two periods; see Caves, Christensen and Diewert (1982; 1409-1411). On index number theory under changing preferences, see also Balk (1989).

[53] See Diewert (2002).

[54] As mentioned earlier, Diewert (1978; 888) showed that all known (at that time) superlative indexes numerically approximated each other to the second order around a point where $p^0 = p^1$ and $q^0 = q^1$. Thus if prices and quantities do not change "too much" between the two periods being compared, P_F, P_W and P_T will generate very similar indexes. It is interesting to note that Edgeworth (1901; 411-412) used the same methodology to show that the Marshall (1887) Edgeworth (1925) index $P_{ME}(p^0,p^1,q^0,q^1) \equiv p^1\cdot(q^0+q^1)/p^0\cdot(q^0+q^1)$ approximated the Walsh index P_W to the second order around an equal price and quantity point.

4. Alternative Approaches to the Specification of Elementary Indexes

4.1. Introduction

In all countries, the calculation of a Consumer Price Index proceeds in two (or more) stages. In the first stage of calculation, *elementary price indexes* are estimated for the *elementary expenditure aggregates* of a CPI. In the second and higher stages of aggregation, these elementary price indexes are combined to obtain higher level indexes using information on the expenditures on each elementary aggregate as weights. An elementary aggregate consists of the expenditures on a small and relatively homogeneous set of products defined within the consumption classification used in the CPI. Samples of prices are collected within each elementary aggregate, so that elementary aggregates serve as strata for sampling purposes.

Data on the expenditures, or quantities, of the different goods and services are typically not available within an elementary aggregate. As there are no quantity or expenditure weights, most of the index number theory outlined in the previous sections is not directly applicable. An elementary price index is a more primitive concept that relies on price data only.

The question of what is the most appropriate formula to use to estimate an elementary price index is considered in this section.[55] The quality of a CPI depends heavily on the quality of the elementary indexes, which are the basic building blocks from which CPIs are constructed.

CPI compilers have to select *representative products* within an elementary aggregate and then collect a sample of prices for each of the representative products, usually from a sample of different outlets. The individual products whose prices are actually collected are described as the *sampled products*. Their prices are collected over a succession of time periods. An elementary price index is therefore typically calculated from two sets of matched price observations. It is assumed initially that there are no missing observations and no changes in the quality of the products sampled so that the two sets of prices are perfectly matched.

4.2. Elementary Indexes used in Practice

Suppose that there are M lowest level items or specific commodities in a chosen elementary category. Denote the period t price of item m by p_m^t for t = 0,1 and for items m = 1,2,...,M. Define the period t price vector as $p^t \equiv [p_1^t,p_2^t,...,p_M^t]$ for t = 0,1.

The first widely used elementary index number formula is due to the French economist Dutot (1738):

$$(38)\quad P_D(p^0,p^1) \equiv [\sum_{m=1}^{M} (1/M)\, p_m^1]/[\sum_{m=1}^{M} (1/M)\, p_m^0] = [\sum_{m=1}^{M} p_m^1]/[\sum_{m=1}^{M} p_m^0].$$

[55] The material in this section draws heavily on the contributions of Dalén (1992), Balk (1994) (1998) (2002) and Diewert (1995a) (2002) which are reflected in the ILO (2004; 355-371).

Thus the Dutot elementary price index is equal to the arithmetic average of the M period 1 prices divided by the arithmetic average of the M period 0 prices.

The second widely used elementary index number formula is due to the Italian economist Carli (1764):

$$(39)\quad P_C(p^0,p^1) \equiv \sum_{m=1}^{M} (1/M)(p_m^1/p_m^0).$$

Thus the Carli elementary price index is equal to the *arithmetic* average of the M item price ratios or price relatives, p_m^1/p_m^0. This formula was already encountered in our study of the unweighted stochastic approach to index numbers; recall (15) above.

The third widely used elementary index number formula is due to the English economist Jevons (1865):

$$(40)\quad P_J(p^0,p^1) \equiv \prod_{m=1}^{M} (p_m^1/p_m^0)^{1/M}.$$

Thus the Jevons elementary price index is equal to the *geometric* average of the M item price ratios or price relatives, p_m^1/p_m^0. Again, this formula was introduced as formula (17) in our discussion of the unweighted stochastic approach to index number theory.

The fourth elementary index number formula P_H is the *harmonic* average of the M item price relatives and it was first suggested in passing as an index number formula by Jevons (1865; 121) and Coggeshall (1887):

$$(41)\quad P_H(p^0,p^1) \equiv [\sum_{m=1}^{M} (1/M)(p_m^1/p_m^0)^{-1}]^{-1}.$$

Finally, the fifth elementary index number formula is the geometric average of the Carli and harmonic formulae; i.e., it is *the geometric mean of the arithmetic and harmonic means of the M price relatives*:

$$(42)\quad P_{CSWD}(p^0,p^1) \equiv [P_C(p^0,p^1)\ P_H(p^0,p^1)]^{1/2}.$$

This index number formula was first suggested by Fisher (1922; 472) as his formula 101. Fisher also observed that, empirically for his data set, P_{CSWD} was very close to the Jevons index, P_J, and these two indexes were his “best’ unweighted index number formulae. In more recent times, Carruthers, Sellwood and Ward (1980; 25) and Dalén (1992; 140) also proposed P_{CSWD} as an elementary index number formula.

Having defined the most commonly used elementary formulae, the question now arises: which formula is “best”? Obviously, this question cannot be answered until desirable properties for elementary indexes are developed. This will be done in a systematic manner in section 4.4 below (using the test approach) but in the present section, one desirable property for an elementary index will be noted. This is the *time reversal test*, which was noted earlier. In the present context, this test for the elementary index $P(p^0,p^1)$ becomes:

(43) $P(p^0,p^1)P(p^1,p^0) = 1$.

This test says that if the prices in period 2 revert to the initial prices of period 0, then the product of the price change going from period 0 to 1, $P(p^0,p^1)$, times the price change going from period 1 to 2, $P(p^1,p^0)$, should equal unity; i.e., under the stated conditions, we should end up where we started.[56] It can be verified that the Dutot, Jevons and Carruthers, Sellwood, Ward and Dalén indexes, P_D, P_J and P_{CSWD}, all satisfy the time reversal test but that the Carli and Harmonic indexes, P_C and P_H, fail this test. In fact, these last two indexes fail the test in the following *biased* manner:

(44) $P_C(p^0,p^1)\ P_C(p^1,p^0) \geq 1$;
(45) $P_H(p^0,p^1)\ P_H(p^1,p^0) \leq 1$

with strict inequalities holding in (44) and (45) provided that the period 1 price vector p^1 is not proportional to the period 0 price vector p^0.[57] Thus the Carli index will generally have an *upward bias* while the Harmonic index will generally have a *downward bias*. As noted earlier, Fisher (1922; 66 and 383) was quite definite in his condemnation of the Carli index due to its upward bias[58] and perhaps as a result, the use of the Carli index was not permitted in compiling elementary price indexes for the Harmonized Index of Consumer Prices (HICP) that is the official Eurostat index used to compare consumer prices across European Union countries.

In the following section, some numerical relationships between the five elementary indexes defined in this section will be established. Then in the subsequent section, a more comprehensive list of desirable properties for elementary indexes will be developed and the five elementary formulae will be evaluated in the light of these properties or tests.

4.3. Numerical Relationships between the Frequently Used Elementary Indexes

It can be shown[59] that the Carli, Jevons and Harmonic elementary price indexes satisfy the following inequalities:

(46) $P_H(p^0,p^1) \leq P_J(p^0,p^1) \leq P_C(p^0,p^1)$;

i.e., the Harmonic index is always equal to or less than the Jevons index which in turn is always equal to or less than the Carli index. In fact, the strict inequalities in (46) will hold

[56] This test can also be viewed as a special case of Walsh's Multiperiod Identity Test, T23.
[57] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901;517) or Fisher (1922; 383-384). This inequality is a special case of Schlömilch's Inequality; see Hardy, Littlewood and Polya (1934; 26).
[58] See also Szulc (1987; 12) and Dalén (1992; 139). Dalén (1994; 150-151) provides some nice intuitive explanations for the upward bias of the Carli index.
[59] Each of the three indexes P_H, P_J and P_C is a mean of order r where r equals –1, 0 and 1 respectively and so the inequalities follow from Schlömilch's inequality; see Hardy, Littlewood and Polya (1934; 26).

provided that the period 0 vector of prices, p^0, is not proportional to the period 1 vector of prices, p^1.

The inequalities (46) do not tell us by how much the Carli index will exceed the Jevons index and by how much the Jevons index will exceed the Harmonic index. Hence, in the remainder of this section, some approximate relationships between the five indexes defined in the previous section will be developed that will provide some practical guidance on the relative magnitudes of each of the indexes.

The first approximate relationship that will be derived is between the Jevons index P_J and the Dutot index P_D. For each period t, define the *arithmetic mean of the M prices* pertaining to that period as follows:

$$(47)\ p^{t*} \equiv \sum_{m=1}^{M} (1/M)\, p_m^t\ ; \qquad t = 0,1.$$

Now define the *multiplicative deviation of the mth price in period t relative to the mean price in that period*, e_m^t, as follows:

$$(48)\ p_m^t = p^{t*}(1+e_m^t)\ ; \qquad m = 1,\ldots,M\ ;\ t = 0,1.$$

Note that (47) and (48) imply that the deviations e_m^t sum to zero in each period; i.e., we have:

$$(49)\ \sum_{m=1}^{M} e_m^t = 0\ ; \qquad t = 0,1.$$

Note that the Dutot index can be written as the ratio of the mean prices, p^{1*}/p^{0*}; i.e., we have:

$$(50)\ P_D(p^0,p^1) = p^{1*}/p^{0*}.$$

Now substitute equations (48) into the definition of the Jevons index, (40):

$$\begin{aligned}(51)\ P_J(p^0,p^1) &= \prod_{m=1}^{M} [p^{1*}(1+e_m^1)/p^{0*}(1+e_m^0)]^{1/M} \\ &= [p^{1*}/p^{0*}] \prod_{m=1}^{M} [(1+e_m^1)/(1+e_m^0)]^{1/M} \\ &= P_D(p^0,p^1)\, f(e^0,e^1) \qquad \text{using definition (38)}\end{aligned}$$

where $e^t \equiv [e_1^t,\ldots,e_M^t]$ for t = 0 and 1, and the function f is defined as follows:

$$(52)\ f(e^0,e^1) \equiv \prod_{m=1}^{M} [(1+e_m^1)/(1+e_m^0)]^{1/M}.$$

Expand $f(e^0,e^1)$ by a second order Taylor series approximation around $e^0 = 0_M$ and $e^1 = 0_M$. Using (49), it can be verified[60] that we obtain the following second order approximate relationship between P_J and P_D:

[60] This approximate relationship was first obtained by Carruthers, Sellwood and Ward (1980; 25).

$$(53)\ P_J(p^0,p^1) \approx P_D(p^0,p^1)[1 + (1/2M)e^0\cdot e^0 - (1/2M)e^1\cdot e^1]$$
$$= P_D(p^0,p^1)[1 + (1/2)\text{var}(e^0) - (1/2)\text{var}(e^1)]$$

where $\text{var}(e^t)$ is the variance of the period t multiplicative deviations; i.e., for t = 0,1:

$$(54)\ \text{var}(e^t) \equiv (1/M)\sum_{m=1}^{M}(e_m{}^t - e^{t*})^2$$
$$= (1/M)\sum_{m=1}^{M}(e_m{}^t)^2 \qquad \text{since } e^{t*} = 0 \text{ using (49)}$$
$$= (1/M)\, e^t\cdot e^t .$$

Under normal conditions[61], the variance of the deviations of the prices from their means in each period is likely to be approximately constant and so under these conditions, the Jevons price index will approximate the Dutot price index to the second order.

Note that with the exception of the Dutot formula, the remaining four elementary indexes defined in subsection 4.2 are functions of the relative prices of the M items being aggregated. This fact is used in order to derive some approximate relationships between these four elementary indexes. Thus define the *mth price relative* as

$$(55)\ r_m \equiv p_m{}^1/p_m{}^0\ ; \qquad m = 1,...,M.$$

Define the *arithmetic mean of the m price relatives* as

$$(56)\ r^* \equiv (1/M)\sum_{m=1}^{M} r_m = P_C(p^0,p^1)$$

where the last equality follows from the definition (39) of the Carli index. Finally, define the *deviation* e_m *of the mth price relative* r_m *from the arithmetic average of the M price relatives* r^* as follows:

$$(57)\ r_m = r^*(1+e_m)\ ; \qquad m = 1,...,M.$$

Note that (56) and (57) imply that the deviations e_m sum to zero; i.e., we have:

$$(58)\ \sum_{m=1}^{M} e_m = 0.$$

Now substitute equations (57) into the definitions of P_C, P_J, P_H and P_{CSWD}, (39)-(42) above, in order to obtain the following representations for these indexes in terms of the vector of deviations, $e \equiv [e_1,...,e_M]$:

$$(59)\ P_C(p^0,p^1) = \sum_{m=1}^{M}(1/M)r_m = r^*1 \equiv r^*f_C(e)\ ;$$
$$(60)\ P_J(p^0,p^1) = \prod_{m=1}^{M} r_m{}^{1/M} = r^*\prod_{m=1}^{M}(1+e_m)^{1/M} \equiv r^*f_J(e)\ ;$$
$$(61)\ P_H(p^0,p^1) = [\sum_{m=1}^{M}(1/M)(r_m)^{-1}]^{-1} = r^*[\sum_{m=1}^{M}(1/M)(1+e_m)^{-1}]^{-1} \equiv r^*f_H(e)\ ;$$

[61] If there are significant changes in the overall inflation rate, some studies indicate that the variance of deviations of prices from their means can also change. Also if M is small, then there will be sampling fluctuations in the variances of the prices from period to period, leading to random differences between the Dutot and Jevons indexes.

(62) $P_{CSWD}(p^0,p^1) = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = r^*[f_C(e)f_H(e)]^{1/2} \quad \equiv r^*f_{CSWD}(e)$

where the last equation in (59)-(62) serves to define the deviation functions, $f_C(e)$, $f_J(e)$, $f_H(e)$ and $f_{CSWD}(e)$. The second order Taylor series approximations to each of these functions[62] around the point $e = 0_M$ are:

(63) $f_C(e) \approx 1$;
(64) $f_J(e) \approx 1 - (1/2M)e{\cdot}e = 1 - (1/2)\mathrm{var}(e)$;
(65) $f_H(e) \approx 1 - (1/M)e{\cdot}e = 1 - \mathrm{var}(e)$;
(66) $f_{CSWD}(e) \approx 1 - (1/2M)e{\cdot}e = 1 - (1/2)\mathrm{var}(e)$

where we have made repeated use of (58) in deriving the above approximations.[63] Thus to the second order, the Carli index P_C will *exceed* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by $(1/2)r^*\mathrm{var}(e)$, which is r^* times one half the variance of the M price relatives p_m^1/p_m^0. Similarly, to the second order, the Harmonic index P_H will *lie below* the Jevons and Carruthers Sellwood Ward Dalén indexes, P_J and P_{CSWD}, by r^* times one half the variance of the M price relatives p_m^1/p_m^0.

Thus empirically, it is expected that the Jevons and Carruthers Sellwood Ward and Dalén indexes will be very close to each other. Using the previous approximation result (53), it is expected that the Dutot index P_D will also be fairly close to P_J and P_{CSWD}, with some fluctuations over time due to changing variances of the period 0 and 1 deviation vectors, e^0 and e^1. Thus it is expected that these three elementary indexes will give much the same numerical answers in empirical applications. On the other hand, the Carli index can be expected to be substantially *above* these three indexes, with the degree of divergence growing as the variance of the M price relatives grows. Similarly, the Harmonic index can be expected to be substantially *below* the three middle indexes, with the degree of divergence growing as the variance of the M price relatives grows.

4.4. The Test Approach to Elementary Indexes

Recall that in subsection 3.4, the axiomatic approach to bilateral price indexes $P(p^0,p^1,q^0,q^1)$ was developed. In the present subsection, the elementary price index $P(p^0,p^1)$ depends only on the period 0 and 1 price vectors, p^0 and p^1 respectively so that the elementary price index does not depend on the period 0 and 1 quantity vectors, q^0 and q^1. One approach to obtaining new tests or axioms for an elementary index is to look at the twenty or so axioms that were listed in subsection 3.4 for bilateral price indexes $P(p^0,p^1,q^0,q^1)$ and adapt those axioms to the present context; i.e., use the old bilateral tests for $P(p^0,p^1,q^0,q^1)$ that do not depend on the quantity vectors q^0 and q^1 as tests for an elementary index $P(p^0,p^1)$.[64] This approach will be utilized in the present subsection.

[62] From (59), it can be seen that $f_C(e)$ is identically equal to 1 so that (63) will be an exact equality rather than an approximation.
[63] These second order approximations are due to Dalén (1992; 143) for the case $r^* = 1$ and to Diewert (1995; 29) for the case of a general r^*.
[64] This was the approach used by Diewert (1995a; 5-17), who drew on the earlier work of Eichhorn (1978; 152-160) and Dalén (1992).

The first eight tests or axioms are reasonably straightforward and uncontroversial:

T1: *Continuity*: $P(p^0,p^1)$ is a continuous function of the M positive period 0 prices $p^0 \equiv [p_1^0,...,p_M^0]$ and the M positive period 1 prices $p^1 \equiv [p_1^1,...,p_M^1]$.

T2: *Identity*: $P(p,p) = 1$; i.e., the period 0 price vector equals the period 1 price vector, then the index is equal to unity.

T3: *Monotonicity in Current Period Prices*: $P(p^0,p^1) < P(p^0,p)$ if $p^1 < p$; i.e., if any period 1 price increases, then the price index increases.

T4: *Monotonicity in Base Period Prices*: $P(p^0,p^1) > P(p,p^1)$ if $p^0 < p$; i.e., if any period 0 price increases, then the price index decreases.

T5: *Proportionality in Current Period Prices*: $P(p^0,\lambda p^1) = \lambda P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 1 prices are multiplied by the positive number λ, then the initial price index is also multiplied by λ.

T6: *Inverse Proportionality in Base Period Prices*: $P(\lambda p^0,p^1) = \lambda^{-1} P(p^0,p^1)$ if $\lambda > 0$; i.e., if all period 0 prices are multiplied by the positive number λ, then the initial price index is multiplied by $1/\lambda$.

T7: *Mean Value Test*: $\min_m\{p_m^1/p_m^0 : m = 1,...,M\} \leq P(p^0,p^1) \leq \max_m\{p_m^1/p_m^0 : m = 1,...,M\}$; i.e., the price index lies between the smallest and largest price relatives.

T8: *Symmetric Treatment of Outlets*: $P(p^0,p^1) = P(p^{0*},p^{1*})$ where p^{0*} and p^{1*} denote the *same* permutation of the components of p^0 and p^1; i.e., if we change the ordering of the outlets (or households) from which we obtain the price quotations for the two periods, then the elementary index remains unchanged.

Eichhorn (1978; 155) showed that Tests 1, 2, 3 and 5 imply Test 7, so that not all of the above tests are logically independent.

The following tests are more controversial and are not necessarily accepted by all price statisticians.

T9: *The Price Bouncing Test*: $P(p^0,p^1) = P(p^{0*},p^{1**})$ where p^{0*} and p^{1**} denote possibly *different* permutations of the components of p^0 and p^1; i.e., if the ordering of the price quotes for both periods is changed in possibly different ways, then the elementary index remains unchanged.

Obviously, T8 is a special case of T9 where the two permutations of the initial ordering of the prices are restricted to be the same. Thus T9 implies T8. Test T9 is due to Dalén (1992; 138). He justified this test by suggesting that the price index should remain unchanged if outlet prices “bounce” in such a manner that the outlets are just exchanging prices with each other over the two periods. While this test has some intuitive appeal, it is not consistent with the idea that outlet prices should be matched to each other in a one to one manner across the two periods.[65]

The following test was also proposed by Dalén (1992) in the elementary index context:

T10: *Time Reversal*: $P(p^1,p^0) = 1/P(p^0,p^1)$; i.e., if the data for periods 0 and 1 are interchanged, then the resulting price index should equal the reciprocal of the original price index.

It is difficult to accept an index that gives a different answer if the ordering of time is reversed.

T11: *Circularity*: $P(p^0,p^1)P(p^1,p^2) = P(p^0,p^2)$; i.e., the price index going from period 0 to 1 times the price index going from period 1 to 2 equals the price index going from period 0 to 2 directly.

The circularity and identity tests imply the time reversal test; (just set $p^2 = p^0$). The circularity property would seem to be a very desirable property: it is a generalization of a property that holds for a single price relative.

T12: *Commensurability*: $P(\lambda_1 p_1^0,..., \lambda_M p_M^0; \lambda_1 p_1^1,..., \lambda_M p_M^1) = P(p_1^0,...,p_M^0; p_1^1,...,p_M^1) = P(p^0,p^1)$ for all $\lambda_1 > 0, ... , \lambda_M > 0$; i.e., if we change the units of measurement for each commodity in each outlet, then the elementary index remains unchanged.

In the bilateral index context, virtually every price statistician accepts the validity of this test. However, in the elementary context, this test is more controversial. If the M items in the elementary aggregate are all very homogeneous, then it makes sense to measure all of the items in the same units. Hence, if we change the unit of measurement in this homogeneous case, then test T12 should restrict all of the λ_m to be the same number (say λ) and test T12 becomes the following test:

$$(67)\quad P(\lambda p^0,\lambda p^1) = P(p^0,p^1) ; \qquad \lambda > 0.$$

Note that (67) will be satisfied if tests T5 and T6 are satisfied.

However, in actual practice, elementary strata will not be very homogeneous: there will usually be thousands of individual items in each elementary aggregate and the hypothesis of item homogeneity is not warranted. Under these circumstances, it is important that the elementary index satisfy the commensurability test, since the units of measurement of the

[65] Since a typical official Consumer Price Index consists of approximately 600 to 1000 separate strata where an elementary index needs to be constructed for each stratum, it can be seen that many strata will consist of quite heterogeneous items. Thus for a fruit category, some of the M items whose prices are used in the elementary index will correspond to quite different types of fruit with quite different prices. Randomly permuting these prices in periods 0 and 1 will lead to very odd price relatives in many cases, which may cause the overall index to behave badly unless the Jevons or Dutot formula is used.

heterogeneous items in the elementary aggregate are arbitrary and hence *the price statistician can change the index simply by changing the units of measurement for some of the items*.

This completes the listing of the tests for an elementary index. There remains the task of evaluating how many tests are passed by each of the five elementary indexes defined in subsection 4.2 above.

It is straightforward to show the following results hold:

- The Jevons elementary index P_J satisfies *all* of the above tests.
- The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails.
- The Carli and Harmonic elementary indexes, P_C and P_H, fail the price bouncing test T9, the time reversal test T10 and the circularity test T11 but pass the other tests.
- The geometric mean of the Carli and Harmonic elementary indexes, P_{CSWD}, fails only the (suspect) price bouncing test T9 and the circularity test T11.

Thus the Jevons elementary index P_J satisfies *all* of the tests and hence emerges as being "best" from the viewpoint of the axiomatic approach to elementary indexes.

The Dutot index P_D satisfies all of the tests with the important exception of the Commensurability Test T12, which it fails. If there are heterogeneous items in the elementary aggregate, this is a rather serious failure and hence price statisticians should be careful in using this index under these conditions.

The geometric mean of the Carli and Harmonic elementary indexes fail only the (suspect) price bouncing test T9 and the circularity test T11. The failure of these two tests is probably not a fatal failure and so this index could be used by price statisticians (who used the test approach for guidance in choosing an index formula), if for some reason, it was decided not to use the Jevons formula. As was noted in subsection 4.3 above, numerically, P_{CSWD} should be very close to P_J.

The Carli and Harmonic elementary indices, P_C and P_H, fail the (suspect) price bouncing test T9, the time reversal test T10 and the circularity test T11 and pass the other tests. The failure of T9 and T11 is again not a fatal failure but the failure of the time reversal test T10 (with an upward bias for the Carli and a downward bias for the Harmonic) is a rather serious failure and so price statisticians should avoid using these indexes.

In the following section, we present an argument due originally to Irving Fisher on why it is desirable for an index number formula to satisfy the time reversal test.

4.5 An Index Number Formula Should be Invariant to the Choice of the Base Period

There is a problem with the Carli and Harmonic indexes which was first pointed out by Irving Fisher:[66] the rate of price change measured by the index number formula between two periods is dependent on which period is regarded as the base period. Thus the Carli index, $P_C(p^0,p^1)$ as defined by (39), takes period 0 as the base period and calculates (one plus) the rate of price change between periods 0 and 1.[67] Instead of choosing period 0 to be the base period, we could equally choose period 1 to be the base period and measure a reciprocal inflation rate going *backwards* from period 1 to period 0 and this *backwards measured inflation rate* would be $\sum_{m=1}^{M} (1/M)(p_m^0/p_m^1)$. In order to make this backwards inflation rate comparable to the forward inflation rate, we then take the reciprocal of $\sum_{m=1}^{M} (1/M)(p_m^0/p_m^1)$ and thus the overall inflation rate going from period 0 to 1 using period 1 as the base period is the following *Backwards Carli index* P_{BC}:[68]

(68) $P_{BC}(p^0,p^1) \equiv [\sum_{n=1}^{N} (1/M)(p_n^1/p_n^0)^{-1}]^{-1} = P_H(p^0,p^1)$;

i.e., the Backwards Carli index turns out to equal the Harmonic index $P_H(p^0,p^1)$ defined earlier by (41).

If the forward and backwards methods of computing price change between periods 0 and 1 using the Carli formula were equal, then we would have the following equality:

(69) $P_C(p^0,p^1) = P_H(p^0,p^1)$.

Fisher argued that a good index number formula should satisfy (69) since the end result of using the formula should not depend on which period was chosen as the base period.[69] This seems to be a persuasive argument: if for whatever reason, a particular formula is favoured, where the base period 0 is chosen to the period which appears before the comparison period 1, then the same arguments which justify the forward looking version of the index number formula can be used to justify the backward looking version. If the forward and backward versions of the index agree with one another, then it does not

[66] "Just as the very idea of an index number implies a set of commodities, so it implies two (and only two) times (or places). Either one of the two times may be taken as the 'base'. Will it make a difference which is chosen? Certainly it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base*." Irving Fisher (1922; 64).

[67] Instead of calculating price inflation between periods 0 and 1, period 1 can be replaced by any period t that follows period 1; i.e., p^1 in the Carli formula $P_C(p^0,p^1)$ can be replaced by p^t and then the index $P_C(p^0,p^t)$ measures price change between periods 0 and t. The arguments concerning $P_C(p^0,p^1)$ which follow apply equally well to $P_C(p^0,p^t)$.

[68] Fisher (1922; 118) termed the backward looking counterpart to the usual forward looking index the *time antithesis* of the original index number formula. Thus P_H is the time antithesis to P_C. The Harmonic index defined by (68) is also known as the Coggeshall (1887) index.

[69] "The justification for making this rule is twofold: (1) no reason can be assigned for choosing to reckon in one direction which does not also apply to the opposite, and (2) such reversibility does apply to any *individual* commodity. If sugar costs twice as much in 1918 as in 1913, then necessarily it costs half as much in 1913 as in 1918." Irving Fisher (1922; 64).

matter which version is used and this equality provides a powerful argument in favour of using the formula. If the two versions do not agree, then rather than picking the forward version over the backward version, a more symmetric procedure would be to take an average of the forward and backward looking versions of the index formula.

Fisher provided an alternative way for justifying the equality of the two indexes in equation (69). He argued that the forward looking inflation rate using the Carli formula is $P_C(p^0,p^1) = \sum_{m=1}^M (1/M)(p_m^1/p_m^0)$. As noted above, the backwards looking inflation rate using the Carli formula is $\sum_{m=1}^M (1/M)(p_m^0/p_m^1) = P_C(p^1,p^0)$. Fisher[70] argued that the product of the forward looking and backward looking indexes should equal unity; i.e., a good formula should satisfy the following equality (which is equivalent to (69)):

(70) $P_C(p^0,p^1)P_C(p^1,p^0) = 1.$

But (70) is the usual *time reversal test* that was listed in the previous section. Thus Fisher provided a reasonably compelling case for the satisfaction of this test.

As we have seen in section 4.3 above,[71] the problem with the Carli formula is that it not only does not satisfy the equalities (69) or (70) but it *fails* (70) with the following definite inequality:

(71) $P_C(p^0,p^1)P_C(p^1,p^0) > 1$

unless the price vector p^1 is proportional to p^0 (so that $p^1 = \lambda p^0$ for some scalar $\lambda > 0$), in which case, (70) will hold. The main implication of the inequality (71) is that the use of *the Carli index will tend to give higher measured rates of inflation* than a formula which satisfies the time reversal test (using the same data set and the same weighting). We will provide a numerical example in section 4.8 below which confirms that this result holds.

Fisher showed how the downward bias in the backwards looking Carli index P_H and the upward bias in the forward looking Carli index P_C could be cured. The Fisher time rectification procedure[72] as a general procedure for obtaining a bilateral index number formula which satisfies the time reversal test works as follows. Given a bilateral price index P, Fisher (1922; 119) defined the *time antithesis* $P°$ for P as follows:

(72) $P^{\circ}(p^0,p^1,q^0,q^1) \equiv 1/P(p^1,p^0,q^1,q^0).$

Thus $P°$ is equal to the reciprocal of the price index which has reversed the role of time, $P(p^1,p^0,q^1,q^0)$. Fisher (1922; 140) then showed that the geometric mean of P and $P°$, say $P^* \equiv [P\times P°]^{1/2}$, satisfies the time reversal test, $P^*(p^0,p^1,q^0,q^1)P^*(p^1,p^0,q^1,q^0) = 1$.

[70] "Putting it in still another way, more useful for practical purposes, the forward and backward index number multiplied together should give unity." Irving Fisher (1922; 64).
[71] Recall the inequalities in (46).
[72] Actually, Walsh (1921b; 542) showed Fisher how to rectify a formula so it would satisfy the factor reversal test and Fisher simply adapted the methodology of Walsh to the problem of rectifying a formula so that it would satisfy the time reversal test.

In the present context, P_C is only a function of p^0 and p^1, but the same rectification procedure works and the time antithesis of P_C is the harmonic index P_H. Applying the Fisher rectification procedure to the Carli index, the resulting rectified Carli formula, P_{RC}, turns out to equal the Carruthers, Sellwood and Ward (1980) and Dalén elementary index P_{CSWD} defined earlier by (42):

$$(73)\ P_{RC}(p^0,p^1) \equiv [P_C(p^0,p^1)P_{BC}(p^0,p^1)]^{1/2} = [P_C(p^0,p^1)P_H(p^0,p^1)]^{1/2} = P_{CSWD}(p^0,p^1).$$

Thus P_{CSWD} is the geometric mean of the forward looking Carli index P_C and its backward looking counterpart $P_{BC} = P_H$, and of course, P_{CSWD} will satisfy the time reversal test.

4.6 A Simple Stochastic Approach to Elementary Indexes

In this section, the Jevons elementary index will be derived using an adaptation of the Country Product Dummy model[73] from the context of comparing prices across two countries to the time series context where the comparison of prices is made between two periods. Suppose that the prices of the M items being priced for an elementary aggregate for periods 0 and 1 are approximately equal to the right hand sides of (74) and (75) below:

$$(74)\ p_m^0 \approx \beta_m\ ; \qquad m = 1,...,M;$$
$$(75)\ p_m^1 \approx \alpha\beta_m\ ; \qquad m = 1,...,M$$

where α is a parameter that can be interpreted as the overall level of prices in period 1 relative to a price level of 1 in period 0 and the β_m are positive parameters that can be interpreted as item specific quality adjustment factors. Note that there are 2M prices on the left hand sides of equations (74) and (75) but only M + 1 parameters on the right hand sides of these equations. The basic hypothesis in (74) and (75) is that the two price vectors p^0 and p^1 are proportional (with $p^1 = \alpha p^0$ so that α is the factor of proportionality) except for random multiplicative errors and hence α represents the underlying elementary price aggregate. If we take logarithms of both sides of (74) and (75) and add some random errors e_m^0 and e_m^1 to the right hand sides of the resulting equations, we obtain the following *linear regression model*:

$$(76)\ \ln p_m^0 = \delta_m + e_m^0; \qquad m = 1,...,M;$$
$$(77)\ \ln p_m^1 = \gamma + \delta_m + e_m^1; \qquad m = 1,...,M$$

where $\gamma \equiv \ln\alpha$ and $\delta_m \equiv \ln\beta_m$ for m = 1,...,M.

---

[73] See Summers (1973) who introduced the CPD model. Balk (1980c) was the first to adapt the CPD method to the time series context.

Note that (76) and (77) can be interpreted as a highly simplified *hedonic regression model* where the δ_m can be interpreted as quality adjustment factors for each item m.[74] The only characteristic of each commodity is the commodity itself. This model is also a special case of the *Country Product Dummy method* for making international comparisons between the prices of different countries. A major advantage of this regression method for constructing an elementary price index is that *standard errors* for the index number α can be obtained. This advantage of the stochastic approach to index number theory was stressed by Selvanathan and Rao (1994).

The least squares estimators for the parameters which appear in (76) and (77) are obtained by solving the following unweighted *least squares minimization problem*:

(78) $\min_{\gamma,\,\delta\text{'s}} \sum_{m=1}^{M} [\ln p_m^0 - \delta_m]^2 + \sum_{m=1}^{M} [\ln p_m^1 - \gamma - \delta_m]^2$.

It can be verified that the least squares estimator for γ is

(79) $\gamma^* \equiv \sum_{m=1}^{M} (1/M)\ln(p_m^1/p_m^0)$.

If γ^* is exponentiated, then the following estimator for the elementary index α is obtained:

(80) $\alpha^* \equiv \prod_{m=1}^{M} [p_m^1/p_m^0]^{1/M} \equiv P_J(p^0,p^1)$

where $P_J(p^0,p^1)$ is the *Jevons elementary price index* defined in section 3.3 above. Thus we have obtained a regression model based justification for the use of the Jevons elementary index.

Although the stochastic model defined by (76) and (77) has not led to a new elementary index number formula (since we just ended up deriving the Jevons index which was already introduced in section 3.3), a generalization of the CPD method adapted to the time series context will be introduced in section 8 below and the present section will serve to introduce the reader to the methods used there.

The following section briefly considers the economic approach to elementary indexes.

4.7. The Economic Approach to Elementary Indexes

The *Consumer Price Index Manual* has a section in it which describes an economic approach to elementary indexes; see the ILO (2004; 364-369). This section has sometimes been used to justify the use of the Jevons index over the use of the Carli index or vice versa depending on how much substitutability exists between items within an elementary stratum. If it is thought that there is a great deal of substitutability between items, then it is suggested that the Jevons index is the appropriate index to use. If it is

[74] For an introduction to hedonic regression models, see Griliches (1971) and Diewert, Heravi and Silver (2009). For an extension of the unweighted CPD model to a situation where information on weights is available, see Balk (1980c), de Haan (2004) and Diewert (2004) (2005) (2006).

thought that there is very little substitutability between items, then it is suggested that the Carli or the Dutot index is the appropriate index to use. This is a misinterpretation of the analysis that is presented in this section of the *Manual*. What the analysis there shows that *if* appropriate sampling of prices can be accomplished over one of the two periods in the comparison, *then* an appropriately probability weighted Carli or Dutot index can approximate a Laspeyres index (which is consistent with preferences that exhibit no substitutability) and an appropriately probability weighted Jevons index can approximate a Cobb-Douglas price index (which is consistent with a Cobb-Douglas subutility function defined over the items in the elementary stratum which has unitary elasticities of substitution). But the appropriate probability weights can only be known *if* knowledge about item quantities purchased is available or *if* information on item expenditures in one period is available. *Such information is typically not available*, which is exactly the reason elementary indexes are used rather than the far superior indexes P_F, P_W or P_T, which require price and quantity information on purchases within the elementary stratum for both periods. *Thus the economic approach cannot be applied at the elementary level unless price and quantity information are both available*.

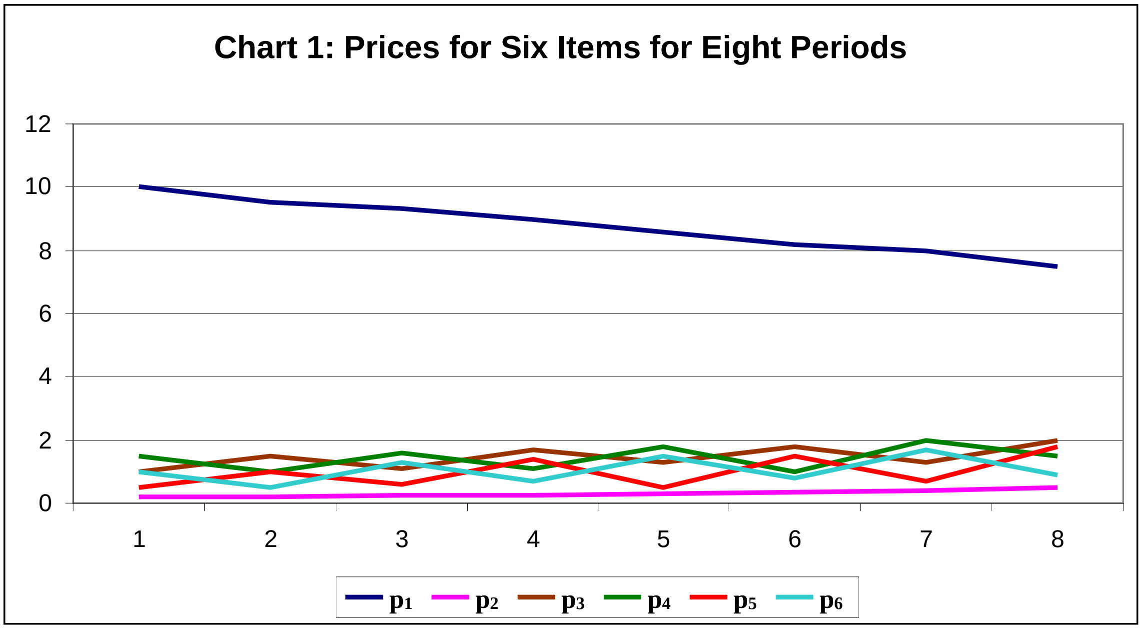
In the following two sections, two numerical examples are provided which illustrate how the various elementary indexes might perform in practice.

4.8 Elementary Indexes: A Numerical Example

In this section, we analyze a small data set consisting of 8 periods of price data for six items. The price of item m in period t is p_m^t for t = 1,...,8 and m = 1,...,6. The data are displayed in Table 1 and Chart 1 below.

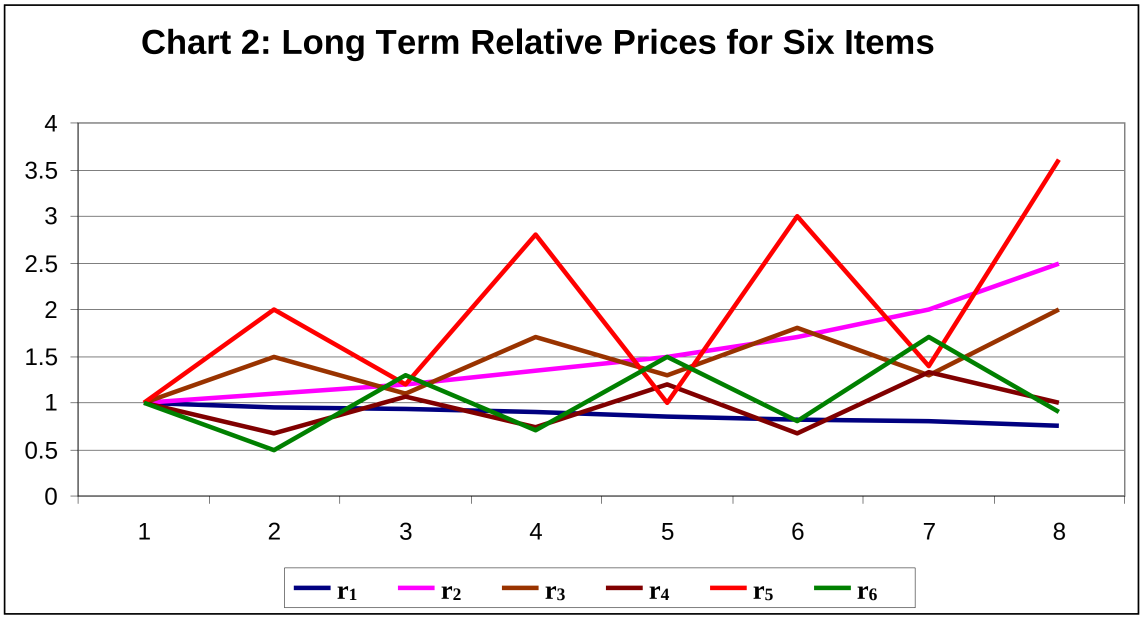
Table 1: Price Data for Six Items for Eight Periods

Period t	p_1^t	p_2^t	p_3^t	p_4^t	p_5^t	p_6^t
1	10.0	0.20	1.0	1.5	0.5	1.0
2	9.5	0.22	1.5	1.0	1.0	0.5
3	9.3	0.24	1.1	1.6	0.6	1.3
4	9.0	0.27	1.7	1.1	1.4	0.7
5	8.6	0.30	1.3	1.8	0.5	1.5
6	8.2	0.34	1.8	1.0	1.5	0.8
7	8.0	0.40	1.3	2.0	0.7	1.7
8	7.5	0.50	2.0	1.5	1.8	0.9



Frequently, an elementary category of goods and services can contain a large number of diverse items. For example, the category "musical instruments" could contain grand pianos, electric guitars and flutes so that some items could have very large prices and some items could have rather small prices. The data in Table 1 reflects such a diverse category: item 1 has very large prices (which trend down smoothly), item 2 has very small prices (which trend up smoothly) while items 3-6 have "average" prices (which bounce around from period to period, reflecting periodic sales of the items). All of the prices have an upward trend, with the exception of item 1.

The ONS computes its Retail Prices Index using item prices over a year relative to the January price of the item. These relative prices are called long term price relatives (as opposed to short term month over month price relatives). For our sample data set, the long term relative prices, $r_n^t \equiv p_n^t/p_n^1$ for n = 1,...,6 and t = 1,...,8, are plotted in Chart 2.



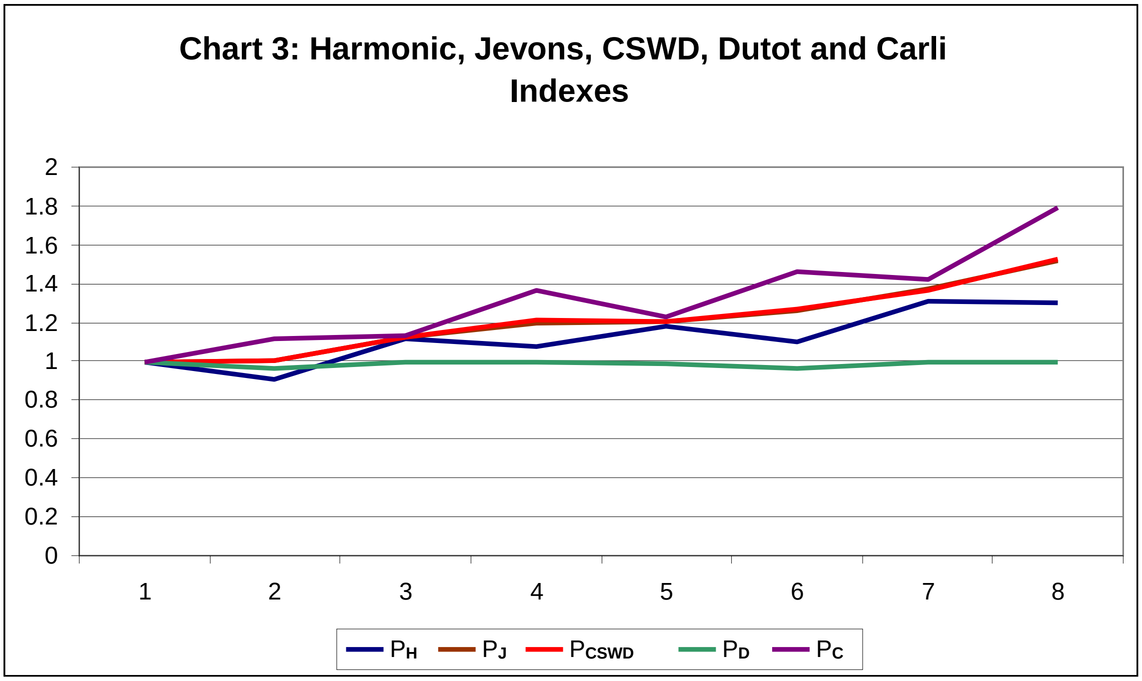
The downward trend in the relative prices for item 1 and the upward trend in the relative prices for item 2 are readily visible in Chart 2 and the price bouncing nature of the long term price relatives for items 3-6 is also visible.

The data in Table 1 are used to calculate the Harmonic, Jevons, Carruthers, Sellwood, Ward and Dalen (CSWD), Dutot and Carli indexes using long term price relatives and the results are listed in Table 2 and plotted in Chart 3 below.[75]

Table 2: Harmonic, Jevons, CSWD, Dutot and Carli Price Indexes Using Long Term Price Relatives for the Artificial Data Set

Period	P_H	P_J	P_{CSWD}	P_D	P_C
1	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.90520	1.00736	1.00664	0.96620	1.11944
3	1.11987	1.12642	1.12631	0.99577	1.13278
4	1.07345	1.19885	1.20999	0.99789	1.36389
5	1.17677	1.20217	1.20146	0.98592	1.22667
6	1.10160	1.26069	1.27013	0.96056	1.46444
7	1.31241	1.36961	1.36621	0.99296	1.42222
8	1.29808	1.51622	1.52503	1.00000	1.79167
Average 2-8	1.14110	1.24020	1.24370	0.98561	1.36020

[75] The index formulae are given by (41), (40), (42), (38) and (39) respectively, except instead of computing $P(p^0,p^1)$, we computed $P(p^1,p^t)$ for the five formulae for t = 1,2,...,8.



As expected, P_J and P_{CWSD} are very close to each other; the two series cannot be distinguished on Chart 3. Also as expected, the Carli index P_C is considerably above the corresponding Jevons and CWSD indexes (for periods 2-8, P_C averages about 12.0 percentage points above the corresponding Jevons index P_J) and the Harmonic index is considerably below the Jevons index (for periods 2-8, P_H averages about 9.9 percentage points below the corresponding Jevons index P_J). Somewhat surprisingly, the Dutot index is far below the other indexes for the later periods (for periods 2-8, P_D averages about 25.5 percentage points below the corresponding Jevons index P_J). This result is due to the very large price for item 1 (and the downward trend in the price of this item): the large price for this item gives the item too much influence in the Dutot index and leads to the anomalous results for this index. However, this example illustrates the problem with the use of the Dutot index in an elementary category: *it will generally not give satisfactory results if the items in the category are quite heterogeneous*.

Recall the results on approximated numerical relationships between the various elementary indexes which were listed in section 4.3 above. It is of some interest to see how well these approximation results are able to predict the differences between the frequently used elementary indexes when applied to the above artificial data set.

We first consider the difference between the Dutot and Jevons indexes. The period t deviations from the mean (the e_m^t for m = 1,...,6 and t = 1,...,T) can be defined by equations (47) and (48). Denote the period t six dimensional vector of deviations as e^t for t = 1,...,8. The adaptation of (53) to the current context leads to the following approximate relationship between P_J and P_D:[76]

[76] Note that $e^t \cdot e^t/6$ is $\mathrm{Var}(e^t)$, the variance of the deviations from the mean in period t for t = 1,...,8. The vector of variances of the deviations is [2.533, 2.426, 2.126, 1.942, 1.793, 1.682, 1.452, and 1.185].

(81) $[P_J(p^1,p^t)/P_D(p^1,p^t)] - 1 \approx (1/2)[e^1\cdot e^1 - e^t\cdot e^t]/6 = (1/2)[\mathrm{Var}(e^1) - \mathrm{Var}(e^t)]$; t = 2,3,...,8.

The arithmetic mean of the seven ratios less unity on the left hand side of (81) is 0.25751 while the average of the seven differences on the right hand side of (81) is 0.36590. Thus the approximation result (53) in section 4.3 is not very accurate for describing the expected difference between the Jevons and Dutot indexes. This lack of accuracy for our numerical example is not too surprising since the approximation results depend on the deviations e_m^t being reasonably small. They are not small in the present case due to the very large prices for item 1 and the very small prices for item 2 relative to the other prices.

However, the numerical approximation results given by (63)-(66) for the remaining four elementary indexes are much more accurate as will be seen, because these approximations use relative prices (rather than price levels, which were used in the Dutot approximations). Recall that the period t price relative to the corresponding period 1 price was defined as $r_m^t \equiv p_m^t/p_m^1$ for t =2,3,...,8 and m = 1,...,6. Adapting definitions (56) and (57) to the current situation, we have the following definitions:

(82) $r^{t*} \equiv (1/6)\sum_{m=1}^{6} r_m^t = P_C(p^1,p^t)$; t = 2,3,...,8 ;
(83) $e_m^t \equiv (r_m^t/r^{t*}) - 1$; m = 1,...,6 ; t = 2,3,...,8.

Let $e^t \equiv [e_1^t,...,e_6^t]$ and let $\mathrm{Var}(e^t)$ denote the variance of the entries in the vector e^t for t = 2,...,8.[77] Then adapting (63)-(66) to the current numerical example, we have the following approximate equalities:

(84) $P_J(p^1,p^t) \approx P_C(p^1,p^t) - (1/2)\mathrm{Var}(e^t)$; t = 2,3,...,8;
(85) $P_{CWSW}(p^1,p^t) \approx P_C(p^1,p^t) - (1/2)\mathrm{Var}(e^t)$; t = 2,3,...,8;
(86) $P_H(p^1,p^t) \approx P_C(p^1,p^t) - \mathrm{Var}(e^t)$; t = 2,3,...,8.

The average variance of the e^t over periods 2-8 is 0.2144 and so one half of this variance is 0.1072. Thus on average (over periods 2-8), we expect the Carli indexes to be 10.7 percentage points above the corresponding Jevons indexes and this compares to the average difference between the two indexes of 12.0 percentage points from Table 2. Similarly we expect the Harmonic indexes to be 10.7 percentage points below the corresponding Jevons indexes and this compares to the average difference between the two indexes of 9.9 percentage points from Table 2. Thus the approximation results listed in section 4.3 above are reasonably accurate for our four indexes that are based on relative prices.

It should be noted that in the above example, there are substantial differences between the Dutot and Jevons indexes, which can be traced to the aggregation of very heterogeneous items (which have different rates of price inflation). There is some evidence that at the aggregate level, there may not be much difference between indexes computed by using

[77] These variances turned out to equal [0.245, 0.013, 0.347, 0.454, 0.375, 0.081, 0.394]. Note that the price bouncing behavior of items 3-6 leads to fairly low variances in some periods.

Jevons elementary indexes versus indexes computed using Dutot elementary indexes. According to Evans (2012), Slovenia uses the Jevons formula exclusively at the elementary levels of their HICP. In their national CPI, however, they exclusively use a Dutot formula. This is because historically the Dutot has always been used in their CPI, and the Statistical Office of the Republic of Slovenia has no plans to change this in the short term. Evans (2012) showed that on average since 1998, the total gap between the Slovenian CPI and HICP has only been 0.1 percentage points.

We have reviewed existing theory dealing with the choice of index number method both when information on prices and quantities is available for two periods (section 3 above) and when only price information is available for two periods (the present section 4). In the following section, we study some of the practical problems that are associated with the construction of a Consumer Price Index.

5. Lowe Indexes and the Practical Construction of a Monthly CPI

5.1 Properties and Alternative Representations for the Lowe Index

Practical index number construction does not proceed exactly along the lines outlined in the previous sections. Even at higher levels of aggregation where expenditure information by commodity class is available, usually this expenditure information comes from household expenditure surveys[78] and there are delays in processing this information. Thus for the RPI, expenditure information that is applied to the 2012 RPI indexes comes from household expenditure data covering the months starting at July 2010 and ending with June 2011.[79] This lack of up to date expenditure information means that all of the approaches to index number theory which were carefully explained in section 3 above cannot be applied if an up to date price index is to be calculated.[80]

Thus practical Consumer Price Indexes use *current price information* at a monthly frequency[81] but they use *quantity or expenditure weights* (at the higher levels of aggregation) that pertain to a past base year. Thus there are *two base periods or reference periods* in a practical CPI: the base year for the reference quantities or expenditures and the base month for prices. This type of index is known in the literature as a *Lowe* (1823) *index*.

It seems a bit odd to use annual quantity or expenditure weights with monthly price information but statistical agencies give two reasons for the use of annual weights:[82]

- Expenditure information collected from household surveys is often unreliable when collected for short periods of time and this variability can be reduced by using annual information;
- Some expenditures are seasonal in nature and thus the pattern of expenditure for any given month will not be representative for the annual average expenditures by commodity class.[83]

[78] In the UK, the household expenditure survey is called the Living Costs and Food Survey (LCF). The survey relates to private households only. For the Retail Prices Index (RPI), the LCF survey expenditure information, excluding highest income households and some pensioner households, is used to obtain weights at higher levels of aggregation. However, for the Consumer Price Index (or CPI which is also the Eurostat mandated HICP), the higher level weights are largely calculated using Household Final Monetary Consumption Expenditure (HFMCE); see the ONS (2012; 35-36).

[79] See the ONS (2012; 41). Evidently, quarterly expenditure information is available in the UK on a more or less continuous basis.

[80] Obviously, the indexes described in section 3 could be calculated on a delayed basis (at least at higher levels of aggregation) if we were willing to wait until expenditure information on the current period became available. If scanner data is used in the construction of a CPI, then the situation changes since up to date information on prices and quantities may be available on a current basis at the detailed item level. Problems associated with consumer price index construction when scanner data are available will be discussed below in section 8.

[81] New Zealand and Australia use current price information at a quarterly frequency.

[82] See the ONS (2012; 12) for these reasons.

We will discuss the above explanations in more detail in section 5.4 below but for now, we will accept the above arguments and proceed to explain alternative ways for representing the Lowe index.

Historically, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ was defined in terms of a base period quantity vector, $q^b \equiv [q_1{}^b,...,q_N{}^b]$ (which we will take to be the base year quantity vector of household purchases), a vector of base month prices p^0 where period 0 represents the base month for pricing purposes and a sequence of 12 consecutive monthly household price vectors $p^t \equiv [p_1{}^t,...,p_N{}^t]$ for m =1,2,...,12 which follow month 0:

$$(87)\ P_{Lo}(p^0,p^t,q^b) \equiv p^t \cdot q^b / p^0 \cdot q^b \equiv \sum_{n=1}^N p_n{}^t q_n{}^b \,/\, \sum_{n=1}^N p_n{}^0 q_n{}^b\ ; \qquad t = 0,1,...,12.$$

Thus the level of prices in month t of the current (augmented) year[84] relative to month 0 is simply the cost of purchasing the commodity basket q^b at the prices p^t of month t, $p^t \cdot q^b$, divided by the cost of purchasing the same annual commodity basket q^b at the prices p^0 of the base month 0, $p^0 \cdot q^b$. This is an index number concept that is relatively easy to explain to the public.

At this point, it will be useful to introduce the notation that corresponds to the type of Lowe index that is used in the monthly Retail Prices Index (RPI) that is constructed by the ONS. The RPI uses January of each year as the base month.[85] Thus if in January of year y, the historical level of the RPI were $P_{RPI}(y{:}1)$, then the February level in year y would be $P_{RPI}(y{:}1) \times P_{Lo}(p^{y,1},p^{y,2},q^b)$, the March level would be $P_{RPI}(y{:}1) \times P_{Lo}(p^{y,1},p^{y,3},q^b)$,..., the December year y level would be $P_{RPI}(y{:}1) \times P_{Lo}(p^{y,1},p^{y,12},q^b)$ and the January year y+1 level would be $P_{RPI}(y+1{:}1) \equiv P_{RPI}(y{:}1) \times P_{Lo}(p^{y,1},p^{y,13},q^b)$. At this point, a new vector of quantity weights would be introduced, say q^{b*} and there would be a new sequence of 13 consecutive monthly price vectors $p^{y+1,m}$ say[86] that started in January of year t+1 and finished in January of year t+2. Thus the February year t+1 level would be $P_{RPI}(y+1{:}1) \times P_{Lo}(p^{y+1,1},p^{y+1,2},q^{b*})$, the March year y+1 level would be $P_{RPI}(y+1{:}1) \times P_{Lo}(p^{0*},p^{2*},q^{b*})$ and so on.

We will rewrite the Lowe index defined by (87) using the notation introduced in the previous paragraph so that p^0 is now replaced by the year y, month 1 vector of prices, $p^{y,1} \equiv [p_1{}^{y,1},p_2{}^{y,1},...,p_N{}^{y,1}]$, and p^t is replaced by the (augmented) year, month m vector of

[83] In fact, the use of the Lowe index $P_{Lo}(p^0,p^t,q^b)$ in the context of seasonal commodities corresponds to Bean and Stine's (1924; 31) Type A index number formula. Bean and Stine made 3 additional suggestions for the construction of price indexes which might be able to deal with seasonal commodities.
[84] An augmented year is a string of 13 consecutive months.
[85] The CPI (or HICP) uses December as the base month in a similar procedure. The CPI index calculations are similar to the RPI index calculations except the link goes from December to January of the following year to meet Eurostat requirements.
[86] If there were no changes in the commodity classification, $p^{y+1,1}$ should equal $p^{y,13}$; i.e., the old January year y+1 vector of prices $p^{y,13}$ should coincide with the first of the new string of 13 consecutive monthly prices $p^{y+1,m}$ for the calendar year y+1 plus January of year y+2.

prices, $p^{y,m} \equiv [p_1^{y,m},p_2^{y,m},...,p_N^{y,m}]$ for m = 1,2,...,13.[87] In what follows, the Lowe index defined above will be written in alternative forms.

The first alternative way of rewriting the sequence of Lowe indexes for year y is in the following *hybrid share form*:

$$(88)\ P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b}) \equiv \sum_{n=1}^{N} p_n^{y,m} q_n^{b} / \sum_{n=1}^{N} p_n^{y,1} q_n^{b} ; \qquad m = 1,2,...,13;$$
$$= \sum_{n=1}^{N} (p_n^{y,m}/p_n^{y,1})p_n^{y,m}q_n^{b} / \sum_{n=1}^{N} p_n^{y,1}q_n^{b}$$
$$= \sum_{n=1}^{N} (p_n^{y,m}/p_n^{y,1})\, s_n^{y,1,b}$$

where the *hybrid expenditure shares* $s_n^{y,1,b}$ corresponding to the (annual) quantity weights vector q^b for the base year b and to the (monthly) prices $p^{y,1}$ for the prices base month (which is January of year y) are defined by:[88]

$$(89)\ s_n^{y,1,b} \equiv p_n^{y,1}q_n^{b} / \sum_{i=1}^{N} p_i^{y,1}q_i^{b} \qquad \text{for } n = 1,\ldots,N.$$

Before proceeding to other representations of the Lowe index, we note that it has very good axiomatic properties *for the comparison of prices within a given augmented year*; i.e., it is straightforward to show that the Lowe index satisfies all of the axioms T1-T12 listed in section 4.4 for elementary indexes except for the following two tests: test T8 (the symmetric treatment of outlets test) and test T9 (the price bouncing test).[89] However, when comparing price vectors across two different augmented years, more tests fail: T2 (the identity test), T7 (the mean value test), T10 (the time reversal test) and T11 (the circularity test).

The failure of the identity test if we compare prices across two different years is particularly troublesome but it is straightforward to find some sufficient conditions that will ensure that this test holds. In order to establish these conditions, it will be necessary to be more explicit on how to convert the present Lowe index methodology (with a fixed January base for an augmented year but then chaining the indexes across years) into indexes that can be put into the format of an elementary index.

Suppose that we start the sequence of index levels at January of year y (and the price level is set equal to unity for this month). Let the sequence of monthly price vectors starting in January of year y and ending in January of year y+1 be denoted by $p^{y,1}$, $p^{y,2}$,..., $p^{y,13}$ and denote the annual base year reference vector of quantities by q^b. Then the sequence of *Lowe index price levels*, $P_{LoL}(y;m)$, *for month m in the augmented year y* will be the following sequence of fixed base indexes:

---

[87] The price vector $p^{y,13}$ is the January of year y+1 price vector where the commodity classification that is used in year y is used to calculate the prices in $p^{y,13}$.

[88] Fisher (1922; 53) used the terminology "weighted by a hybrid value" while Walsh (1932; 657) used the term "hybrid weights". The two representations of the Lowe index given by the first and last lines of equations (88) are given in the ONS (2012; 12)

[89] The test T8 and T9 are not really relevant in the present context. The representation of the Lowe index given by (88) and (89) is useful in establishing that the Lowe index satisfies test T7, the mean value test in section 4.4.

(90) $P_{LoL}(y;m) \equiv P_{Lo}(p^{y,1},p^{y,m},q^b) = p^{y,m}\cdot q^b/p^{y,1}\cdot q^b$; m = 1,2,...,13.

The elementary index tests in section 4.4 involve two price vectors: a base period price vector p^0 and a current period price vector p^1 and a function, $P(p^0,p^1)$, We will redefine the base period price vector p^0 as $p^{y,r}$ and the current period price vector p^1 as $p^{y,s}$ where r and s refer to two months in the augmented year y. As usual, q^b is the annual basket vector that is used to compute Lowe indexes for year y. The "elementary" price index $P(p^{y,r},p^{y,s})$ that corresponds to the comparison of the Lowe index price level in month s relative to month r in the augmented year y is defined as follows:

$$
\begin{aligned}
(91)\ P(p^{y,r},p^{y,s}) &\equiv P_{Lo}(p^{y,1},p^{y,s},q^b)/P_{Lo}(p^{y,1},p^{y,r},q^b) \\
&= [p^{y,s}\cdot q^b/p^{y,1}\cdot q^b]/[p^{y,r}\cdot q^b/p^{y,1}\cdot q^b] \\
&= p^{y,s}\cdot q^b/p^{y,r}\cdot q^b.
\end{aligned}
$$

It can be verified that $P(p^{y,s},p^{y,r}) \equiv p^{y,s}\cdot q^b/p^{y,r}\cdot q^b$ passes the tests T1-T7 and T10-T12 listed in section 4.4.

Now suppose that we wish to compare the prices in month s of year y+1 with the prices of month r in year y. We first construct the sequence of *price levels* for months m in augmented year y+1, $P_{LoL}(y+1;m)$. Let the sequence of monthly price vectors starting in January of year y+1 and ending in January of year y+2 be denoted by $p^{y+1,1}$, $p^{y+1,2}$,..., $p^{y+1,13}$ and denote the new annual base year reference vector of quantities for year y+1 by q^{b+1}.[90] The January level of the price index for year y+1 has already been determined as $P_{LoL}(y;12) \equiv p^{12}\cdot q^b/p^0\cdot q^b$; see (90) with m = 12. The sequence of *Lowe index price levels*, $P_L(y+1;m)$, *for month m in the augmented year y+1* will be the product of the already determined January value of the fixed base Lowe index for month 12 in year t, $P_{Lo}(p^{y,1},p^{y,13},q^b) = P_{LoL}(y,13)$ times the new fixed base Lowe index for month m in augmented year y+1, $P_{Lo}(p^{y+1,1},p^{y+1,m},q^{b+1})$; i.e., we have the following sequence of *price levels for augmented year y+1*:

$$
\begin{aligned}
(92)\ P_{LoL}(y+1;m) &\equiv P_{LoL}(y,13)P_{Lo}(p^{y+1,1},p^{y+1,m},q^{b+1}) \qquad m = 1,2,...,13 \\
&= [p^{y,13}\cdot q^b/p^{y,1}\cdot q^b][p^{y+1,m}\cdot q^{b+1}/p^{y+1,1}\cdot q^{b+1}] ;
\end{aligned}
$$

Now let $p^{y,r}$ and $p^{y+1,s}$ be the price vectors for month r in augmented year y and for month s in augmented year y+1. The "elementary" price index $P(p^{y,r},p^{y+1,s})$ that compares the month s prices in the vector $p^{y+1,s}$ for year y+1 relative to the month r prices in the vector $p^{y,r}$ for year is defined as the ratio of the price level in month s of year y+1, $P_{LoL}(y+1;s)$, relative to the price level in month r of year y, $P_{LoL}(y;r)$:

$$
\begin{aligned}
(93)\ P^*(p^{y,r},p^{y+1,s}) &\equiv P_{LoL}(y+1;s)/P_{LoL}(y;r) \\
&= [p^{y,13}\cdot q^b/p^{y,1}\cdot q^b][p^{y+1,s}\cdot q^{b+1}/p^{y+1,1}\cdot q^{b+1}]/[p^{y,r}\cdot q^b/p^{y,1}\cdot q^b] \text{ using (90) and (92)}
\end{aligned}
$$

[90] If the commodity classification remains the same in years y and y+1, then $p^{y,13}$ will equal $p^{y+1,1}$. However, typically, there will be small changes in the commodity classifications going from one year to the next and these changes may make $p^{y,13} \neq p^{y+1,1}$.

$$= [p^{y+1,s}\cdot q^{b+1}/p^{y+1,1}\cdot q^{b+1}]/[p^{y,r}\cdot q^{b}/p^{y,13}\cdot q^{b}].$$

Thus $P^*(p^{y,r},p^{y+1,s})$, which compares the level of prices in month s of year y+1 to the level of prices in month r of year y, is equal to the year y+1 Lowe index $p^{y+1,s}\cdot q^{b+1}/p^{y+1.1}\cdot q^{b+1}$ (which compares the prices in month s to month 0 (January) in year y+1) divided by the year y Lowe index $p^{y,r}\cdot q^{b}/p^{y,13}\cdot q^{b}$ (which compares the prices $p^{y,r}$ in month r in year y to the prices $p^{y,13}$ in January of year y+1, using the year y classification scheme).

It can be verified that the "elementary" index P^* defined by (93) satisfies the following tests listed in section 4.4 above: T1 (continuity), T3 (monotonicity in the components of $p^{y+1,s}$), T4 (monotonicity in the components of $p^{y,r}$), T5 (homogeneity of degree 1 in the components of $p^{y+1,s}$), T6 (homogeneity of degree −1 in the components of $p^{y,r}$), and T12 (commensurability). The following tests are not in general satisfied by P^*: T2 (identity), T7 (mean value test), T8 (symmetric treatment of outlets test), T9 (price bouncing test), T10 (time reversal test) and T11 (circularity). The failures of the identity test and the time reversal test are fairly serious.

However, if we make some extra assumptions, the "elementary" index defined by (93) simplifies and satisfies more tests. The extra assumptions are listed in (94):

(94) $p^{y,13} = p^{y+1,1}$ and $q^{b} = q^{b+1.}$

If there are no changes in the commodity classification, then $p^{y,13}$ (the vector of January prices for year y+1 using the classification scheme of year y) will indeed be equal to $p^{y+1,1}$ (the vector of January prices for year t+1 using the classification scheme of year y+1) so the first equality in (94) will hold. If the reference annual quantity vector for year b, q^{b}, happens to remain unchanged for the following year b+1, then $q^{b} = q^{b+1}$ and the second equality in (94) will hold. Thus if conditions (94) hold, then (93) simplifies as follows:

(95)$P^*(p^{y,r},p^{y+1,s}) = p^{y+1,s}\cdot q^{b}/p^{y,r}\cdot q^{b}$.

The "elementary" index P^* defined by (95) will satisfy all of the tests T1-T12 with the exception of tests T8 and T9.

Generally speaking, $p^{y,13}$ will be close to $p^{y+1,1}$ so the first equality in (94) will be approximately true. The second equality in (94) will also be approximately true (household purchases of commodities do not change all that much going from one year to the next) but the degree of approximation will not be as close as the closeness of the two price vectors. In particular, if there are long run trends in prices, then there will be substitution effects that will cause q^{b+1} to systematically diverge from q^{b}.

From the economic perspective, the *Consumer Price Index Manual* showed that if there are long term trends in prices, then the Lowe index was likely to have some substitution bias;[91] however, since the ONS updates its weights every year for both the RPI and the HICP, this substitution bias is likely to be small.

In order to provide the additional alternative characterizations of the Lowe index (within an augmented year), it is necessary to introduce *base year expenditures by commodity*, say e_n^b for n = 1,...,N, and *base year average prices by commodity*, say p_n^b for n = 1,...,N. Of course, average annual prices, annual quantities and annual expenditures should satisfy the following equations:

$$(96)\ e_n^b = p_n^b q_n^b\ ; \qquad n = 1,...,N.$$

Now let $p^b \equiv [p_1^b,...,p_N^b]$ and rewrite the sequence of within the (augmented) year Lowe indexes defined by (88) as follows:

$$\begin{aligned}(97)\ P_{Lo}(p^{y,1},p^{y,m},q^b) &= p^{y,m}\cdot q^b/p^{y,1}\cdot q^b\ ; \qquad m = 1,2,...,13;\\ &= [p^{y,m}\cdot q^b/p^b\cdot q^b]/[p^{y,1}\cdot q^b/p^b\cdot q^b]\\ &= \textstyle\sum_{n=1}^N s_n^b(p_n^{y,m}/p_n^b)/\sum_{n=1}^N s_n^b(p_n^{y,1}/p_n^b)\end{aligned}$$

where the *base year expenditure shares* s_n^b are defined as

$$(98)\ s_n^b \equiv p_n^b q_n^b/\textstyle\sum_{i=1}^N p_i^b q_i^b = e_n^b/\sum_{i=1}^N e_i^b\ ; \qquad n = 1,...,N.$$

The Laspeyres index between two price vectors p^0 and p^1 can be defined as $P_L(p^0,p^1,q^0) \equiv p^1\cdot q^0/p^0\cdot q^0$; recall (7) above. Thus using (97), it can be seen that the Lowe index can be written as the ratio of the Laspeyres index $P_L(p^b,p^{y,m},q^b)$ that compares the prices of month m in year y, $p^{y,m}$, to the base year prices p^b and the Laspeyres index $P_L(p^b,p^{y,1},q^b)$ that compares the prices of month 1 in year y, $p^{y,1}$, to the base year prices p^b:[92]

$$\begin{aligned}(99)\ P_{Lo}(p^{y,1},p^{y,m},q^b) &= [p^t\cdot q^b/p^b\cdot q^b]\,/[p^0\cdot q^b/p^b\cdot q^b] \qquad m = 1,2,...,13;\\ &= P_L(p^b,p^{y,m},q^b)/P_L(p^b,p^{y,1},q^b).\end{aligned}$$

It is useful to explain how the annual price and quantity vectors, p^b and q^b, can be obtained from monthly price and expenditure data on each commodity during the chosen base year b. Let $p_n^{b,m}$ be the *monthly (unit value) price for commodity n in month m of the base year b* and let $e_n^{b,m}$ be the corresponding *monthly expenditure for the reference population for commodity n in month m of the base year b* for n = 1,...,N and m = 1,2,...,12. The *annual total consumption for commodity n for base year b for the reference population*, q_n^b, can be obtained by deflating monthly values and summing over months in the base year b as follows:

$$(100)\ q_n^b \equiv \textstyle\sum_{m=1}^{12} e_n^{b,m}/p_n^{b,m} = \sum_{m=1}^{12} q_n^{b,m}\ ; \qquad n = 1,\ldots,N$$

[91] See ILO (2004; 273).
[92] This formula for the Lowe index can be found in ILO (2004; 271).

where $q_n^{b,m} \equiv e_n^{b,m}/p_n^{b,m}$ for n = 1,...,N and m = 1,...,12. In practice, the above equations will be evaluated using aggregate expenditures over closely related commodities and the price $p_n^{b,m}$ will be the month m price index for this elementary commodity group n in year b relative to the first month of year b.[93]

Following national income accounting conventions, a reasonable[94] *annual price for commodity n for the base year b*, p_n^b, which matches up with the annual quantity q_n^b defined by (100) is the *value* of total consumption of commodity n in year b divided by q_n^b. Thus we have:

$$(101)\ p_n^b \equiv (\textstyle\sum_{m=1}^{12} e_n^{b,m})/q_n^b \qquad n = 1,\ldots,N$$
$$= \textstyle\sum_{m=1}^{12} e_n^{b,m}/[\sum_{m=1}^{12} e_n^{b,m}/p_n^{b,m}] \qquad \text{using (100)}$$
$$= [\textstyle\sum_{m=1}^{12} s_n^{b,m}(p_n^{b,m})^{-1}]^{-1}$$

where the *share of annual expenditure on commodity n in month m of the base year* is

$$(102)\ s_n^{b,m} \equiv e_n^{b,m}/\textstyle\sum_{k=1}^{12} e_n^{b,k}\ ; \qquad n = 1,\ldots,N\ ;\ m = 1,\ldots,12.$$

Thus the annual base year price for commodity n, p_n^b, turns out to be a monthly expenditure weighted *harmonic mean* of the monthly prices for commodity n in the base year, $p_n^{b,1}, p_n^{b,2},\ldots, p_n^{b,12}$.

Once the base year prices p_n^b have been calculated, the hybrid shares s_n^{0b} defined by (89) can be calculated by multiplying the base year expenditures e_n^b by $(p_n^{y,1}/p_n^b)$, the ratio of the January price for commodity n in year y, $p_n^{y,1}$ to the base year price for commodity n, p_n^b. Thus the nth hybrid share can be written as follows:

$$(103)\ s_n^{y,1,b} \equiv p_n^{y,1}q_n^b / \textstyle\sum_{i=1}^N p_i^{y,1}q_i^b = (p_n^{y,1}/p_n^b)e_n^b/\sum_{k=1}^N (p_k^{y,1}/p_k^b)e_k^b.$$

The operation of multiplying the base year expenditure weight for commodity n, e_n^b, by the corresponding nth price ratio, $p_n^{y,1}/p_n^b$, is known as *price updating* the base year expenditure weights. Substitution of (103) into the last line of (88) leads to our *third formula*[95] for the Lowe index:

$$(104)\ P_{Lo}(p^{y,1},p^{y,m},p^b,e^b) = \textstyle\sum_{n=1}^N (p_n^{y,m}/p_n^{y,1})(p_n^{y,1}/p_n^b)e_n^b/\sum_{k=1}^N (p_k^{y,1}/p_k^b)e_k^b;\ m = 1,2,...,13$$

[93] A further complication is that at present, the ONS does not have access to monthly expenditure information by commodity group for the target population: only quarterly estimates are available. Thus in order to apply the above algebra, it will be necessary for the ONS to generate monthly expenditure estimates. These monthly estimates do not have to be very accurate: rough approximations will suffice.

[94] Hence these annual commodity prices are essentially unit value prices. Under conditions of high inflation, the annual prices defined by (6) may no longer be “reasonable” or representative of prices during the entire base year because the expenditures in the final months of the high inflation year will be somewhat artificially blown up by general inflation. Under these conditions, the annual prices and annual commodity expenditure shares should be interpreted with caution. For more on dealing with situations when there is high inflation within a year, see Hill (1996).

[95] The first two formulae are (87) and (88).

where $e^b \equiv [e_1^b,...,e_N^b]$ is the vector of base year expenditure shares on the N commodities. The above formula shows how the Lowe indexes are functions of four sets of variables: $p^{y,1}$ (the month 1 price vector for year y),$p^{y,m}$ (the month m price vector for year y), p^b (the vector of annual commodity prices for the base year b) and e^b (the vector of annual household expenditures for the reference population in year b).[96]

Formula (104) for the Lowe index leads directly to our final formula for this index. Thus divide both numerator and denominator on the right hand side of (104) by total annual expenditures by the reference population in the base year b, $\sum_{n=1}^N e_n^b$, and using definitions (102) which define the annual expenditure shares s_n^b for the base year, it can be seen that the Lowe index can be written as the *ratio of two Young* (1812) *indexes*:

$$(105)\ P_{Lo}(p^{y,1},p^{y,m},p^b,s^b) = \sum_{n=1}^N (p_n^{y,m}/p_n^b)s_n^b/\sum_{k=1}^N (p_k^{y,1}/p_k^b)s_k^b; \qquad m = 1,2,...,13$$
$$= P_Y(p^b,p^{y,m},s^b)/P_Y(p^b,p^{y,1},s^b)$$

where the *Young index* $P_Y(p^0,p^1,s)$ which compares the prices p^1 to the prices p^0 using the share weights s is defined as

$$(106)\ P_Y(p^0,p^1,s) \equiv \sum_{n=1}^N (p_n^1/p_n^0)s_n.$$

Comparing the Young index defined by (106) with the Carli index defined by (39), it can be seen that the Carli index is a special case of the Young index when the weights s_n are all equal to 1/N. Unfortunately, the Young index has the same upward bias problem that made the Carli index an unattractive choice of elementary index; i.e., it can be shown that the Young index fails the time reversal test with the likelihood of an upward bias. In particular, the following inequality holds which is the counterpart to the inequality (44) for the Carli index:[97]

$$(107)\ P_Y(p^0,p^1,s)P_Y(p^1,p^0,s) \geq 1$$

where the strict inequality in (107) holds unless p^1 is proportional to p^0.

However, since the Lowe index is a ratio of Young indexes, there is no obvious bias in an index that is equal to a ratio of Young indexes, as in (105).

Having introduced the concept of a Young index, it is useful to contrast the following Young index, $P_Y(p^{y,1},p^{y,m},s^b)$, which compares the prices in month m of year y, $p^{y,m}$, to the prices of month 1 in year m, $p^{y,1}$, using the base year expenditure shares s^b as weights:

$$(108)\ P_Y(p^{y,1},p^{y,m},s^b) \equiv \sum_{n=1}^N (p_n^{y,m}/p_n^{y,1})\, s_n^b\ ; \qquad m = 1,2,...,13.$$

[96] Actually, it can be seen that the Lowe index depends only on three vectors: the *within year y vector of relative prices* $[p_1^{y,m}/p_1^{y,1},p_2^{y,m}/p_2^{y,1},...,p_N^{y,m}/p_N^{y,1}]$ that compares the prices in month m of year y with the corresponding commodity prices of month 1 of year y; the *vector of relative prices* $[p_1^{y,1}/p_1^b,...,p_N^{y,1}/p_N^b]$ that compares the prices in month 1 of year y with the corresponding commodity prices of the base year and the *annual expenditures vector for the base year* b, $e^b \equiv [e_1^b,...,e_N^b]$.

[97] See the ILO (2004; 277).

The above Young index can be compared to the representation of the corresponding Lowe index given by (88), $P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b})$, which compared the same monthly price vectors, $p^{y,1}$ and $p^{y,m}$ in year m, but used the hybrid expenditure shares $s^{y,1,b}$ defined by (89) as weights in place of the base year expenditure shares s^b:

$$(109)\ P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b}) \equiv \sum_{n=1}^{N} (p_n^{y,m}/p_n^{y,1})\, s_n^{y,1,b}\ ; \qquad m = 1,2,...,13.$$

The hybrid expenditure shares, $s_n^{y,1,b}$, can be regarded as price updated versions of the base year expenditure shares; i.e., starting with definitions (89), we have the following representation for the hybrid shares:

$$\begin{aligned}(110)\ s_n^{y,1,b} &\equiv p_n^{y,1}q_n^b / \textstyle\sum_{i=1}^{N} p_i^{y,1}q_i^b \qquad \text{for } n = 1,\ldots,N\\ &= (p_n^{y,1}/p_n^b)p_n^bq_n^b / \textstyle\sum_{i=1}^{N} (p_i^{y,1}/p_i^b)p_i^bq_i^b\\ &= (p_n^{y,1}/p_n^b)e_n^b / \textstyle\sum_{i=1}^{N} (p_i^{y,1}/p_i^b)e_i^b\\ &= (p_n^{y,1}/p_n^b)s_n^b / \textstyle\sum_{i=1}^{N} (p_i^{y,1}/p_i^b)s_i^b\end{aligned}$$

where the last equality follows by dividing numerator and denominator by the base year annual expenditures, $\sum_{n=1}^{N} e_n^b$, and using definitions (98) which define the base year expenditure shares, s_n^b.

At first glance, it would appear that the indexes defined by (108) and (109) should be numerically close. Obviously, if inflation is uniform across all commodity classes going from the base year to the base month 1 in year y, so that $p_n^{y,1} = \lambda p_n^b$ for n = 1,...,N for some scalar $\lambda > 0$, then $s_n^{y,1,b}$ will equal s_n^b for all n with the consequence that the Young index $P_Y(p^{y,1},p^{y,m},s^b)$ defined by (108) will equal the Lowe index defined by (109). However, this price proportionality assumption is unlikely to hold so we will develop a more general necessary and sufficient condition for the equality of the Lowe and Young indexes defined by (108) and (109).

In order to simplify the notation, define the *relative price* r_n *between months m and 1 for commodity n in year y* and the *relative price* t_n *between month 1 in year y for commodity n relative to the average price of commodity n in the base year b* as follows:

$$(111)\ r_n \equiv p_n^{y,m}/p_n^{y,1}\ ;\ t_n \equiv (p_n^{y,1}/p_n^b)\ ; \qquad n = 1,...,N.$$

Define r^* as the *share weighted average of the* r_n and t^* as the *share weighted average of the* t_n, where the base year expenditure shares, s_n^b, are used as weights as follows:

$$(112)\ r^* \equiv \sum_{n=1}^{N} s_n^b r_n = P_Y(p^{y,1},p^{y,m},s^b)\ ;\ t^* \equiv \sum_{n=1}^{N} s_n^b t_n = P_Y(p^b,p^{y,1},s^b).$$

Note that r^* is equal to the Young index $P_Y(p^{y,1},p^{y,m},s^b)$ defined by (108). It will also be useful to define the *weighted covariance between the relative price vectors* $r \equiv [r_1,...,r_N]$ and $t \equiv [t_1,...,t_N]$ using the base year shares s_n^b as weight as follows:

$$(113)\ \mathrm{Cov}(r,t,s^b) \equiv \sum_{n=1}^{N} (r_n-r^*)(t_n-t^*)s_n^b = \sum_{n=1}^{N} r_n t_n s_n^b - r^*t^*.$$

Substituting definitions (111) into (109) leads to the following expression for the Lowe index $P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b})$:

$$(113)\ P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b}) \equiv \sum_{n=1}^{N} r_n s_n^{y,1,b}\ ; \qquad m = 1,2,...,13$$
$$= \sum_{n=1}^{N} r_n t_n s_n^{b}/\sum_{n=1}^{N} t_n s_n^{b} \qquad \text{using (110) and (111)}$$
$$= [\mathrm{Cov}(r,t,s^b) + r^* t^*]/t^* \qquad \text{using (112) and (113)}$$
$$= [\mathrm{Cov}(r,t,s^b)/t^*] + P_Y(p^{y,1},p^{y,m},s^b) \qquad \text{using (112).}$$

Thus we obtain the following simple relationship between the Young index $P_Y(p^{y,1},p^{y,m},s^b)$ which uses the base year shares s_n^b as weights for the relative prices $p_n^{y,m}/p_n^{y,1}$ and the corresponding Lowe index $P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b})$ which uses the hybrid share vector $s_n^{y,1,b}$ as weights:

$$(114)\ P_{Lo}(p^{y,1},p^{y,m},s^{y,1,b}) - P_Y(p^{y,1},p^{y,m},s^b) = \mathrm{Cov}(r,t,s^b)/t^* = \mathrm{Cov}(r,t,s^b)/P_Y(p^b,p^{y,1},s^b)$$

where the last equality follows using (112). Thus the difference between the Lowe index for month m in year y (that uses the price updated hybrid shares $s_n^{y,1,b}$ as weights for the price relatives $p_n^{y,m}/p_n^{y,1}$) and the corresponding Young index for month m in year y (that uses the base year expenditure shares s_n^b as weights for the price relatives $p_n^{y,m}/p_n^{y,1}$) is equal to the weighted covariance between the within year y price relatives $r_n \equiv p_n^{y,m}/p_n^{y,1}$ and the price relatives $t_n \equiv p_n^{y,1}/p_n^b$ between the base year and month 1 of year y, $\mathrm{Cov}(r,t,s^b)$, divided by the Young index $P_Y(p^b,p^{y,1},s^b)$, which measures price inflation going from the base year b to month 1 of the current year y.

Since $P_Y(p^b,p^{y,1},s^b)$ will generally be a number which is slightly larger than 1, the key term which will explain the difference between the Lowe and Young indexes is the covariance, $\mathrm{Cov}(r,t,s^b)$. If price change over all commodity groups proceeds smoothly with long run trends in most strata, then this covariance will be positive and the Lowe index will exceed the corresponding Young index. However, if there is mean reversion of prices (so that a relatively high average price p_n^b for commodity n in the base year is followed by relatively low monthly prices $p_n^{y,m}$ for this commodity in the current year y), then the covariance will be negative.[98] The situation is also complicated by the existence of seasonality in the monthly prices for year y; this seasonality could cause the covariance $\mathrm{Cov}(r,t,s^b)$ to be either positive or negative.[99]

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[98] This appears to be the case for computations based on some Israeli data on fresh fruits. Diewert, Finkel and Artsev (2009) computed Lowe and Young indexes for fresh fruits for the 72 months starting in January 1997 and extending through December 2002 and the sample means of the Lowe and Young indexes were 1.1220 and 1.1586 respectively.

[99] Our discussion of the use of Young indexes in the place of Lowe indexes is not irrelevant to the ONS situation. At lower levels of aggregation, the ONS uses replication weights that are not price updated and so their Lowe type index is not a "pure" Lowe index but rather has some elements of Young indexes in their procedures; see the ONS (2012; 38-41). The above algebra suggests that if $\mathrm{Cov}(r,t,s^b)$ is small in magnitude for the stratum under consideration, then price updating the weights at lower levels of aggregation will not materially affect the overall index.

Our conclusion at this point is that the Lowe index is a reasonably satisfactory index concept for the construction of a practical consumer price index. In particular, its axiomatic properties are reasonably satisfactory. However, these axioms do not deal adequately with seasonal baskets and so later in this report, we will suggest indexes that deal more adequately with seasonal commodities. Since the ONS updates its annual expenditure weights on a continuous basis, the substitution bias that is inherent in a fixed basket index will be relatively low under normal conditions.[100] However as noted above, there are some problems with the use of annual weights that are used in conjunction with monthly prices and these problems will be discussed later in section 5.3. In addition, there are some specific problems with ONS procedures that will be discussed in the following section.

5.2 Problems with the Estimation of Annual Prices and Quantities for the Base Year

The algebra in the previous section implicitly assumed that the ONS collected price and expenditure data for every distinct product that is sold to households in the UK over the course of a year. This is an oversimplification: expenditures are split up into strata and within each stratum, specific products within the strata are chosen to be priced. The underlying assumption is that the sampled specific product prices capture the trend for all products in the strata. At present, the ONS collects item prices for 705 specific products but these collected prices are generally stratified by location (12 regions of the UK are distinguished) and by shop type. In the end, expenditure weights for 5000-5500 strata are constructed. At the start of each calendar year, the annual expenditure weights refer to a “split” year consisting of the first 6 months of the previous year and the last six months of the calendar year of two years ago. With respect to prices, approximately 160,000 item prices are collected each month from various retail outlets (or centrally for some commodities).

At higher levels of aggregation (i.e., at the level of the 5000 or so strata for which there is expenditure weight information), the ONS uses the Lowe index as its target index concept. However, note that within each stratum for which expenditure weights are available, an elementary price index is used in place of true micro prices for each product in each stratum. The problems associated with the choice of elementary aggregate formula were reviewed in more detail in section 4. In this section, we will look at how the ONS price updates its annual expenditure information.

The ONS does not follow the Lowe index methodology as outlined in the previous section. In particular, it uses a modification of formula (104) to compute the RPI, where we now interpret $p_n^{b,m}$ as the elementary price index for stratum n for month m of base year b and $p_n^{y,m}$ as the same elementary price index for stratum n but for month m of the current year y. Recall the price updating methodology explained in the previous section where the base year expenditure weight for commodity group n, e_n^b, was multiplied by the corresponding nth price ratio, $p_n^{y,1}/p_n^b$, where the annual price for stratum n in the

[100] For the 2012 ONS RPI, upper level expenditure weights are based on household expenditure data covering the period starting in July 2010 and ending in June 2011. Thus at the start of 2012, the expenditure weights are only 6 months out of date; see the ONS (2012; 41).

base year b, p_n^b, turned out to equal the share weighted harmonic average of the monthly prices in the base year b, $[\sum_{m=1}^{12} s_n^{b,m} (p_n^{b,m})^{-1}]^{-1}$. The ONS (2012; 41) method for updating the base year expenditure weights for the RPI uses the elementary price indexes for the previous January, $p_n^{y-1,1}$, in place of base year average prices, p_n^b, defined by (101). Thus in place of the "true" Lowe indexes defined by (104), $P_{Lo}(p^{y,1},p^{y,m},p^b,e^b)$, the ONS uses the following *approximate Lowe indexes* when constructing their RPI:

$$(115)\ P_{LoRPI}(p^{y,1},p^{y,m},p^{y-1,1},e^b) = \sum_{n=1}^N (p_n^{y,m}/p_n^{y,1})(p_n^{y,1}/p_n^{y-1,1})e_n^b/\sum_{k=1}^N (p_k^{y,1}/p_k^{y-1,1})e_k^b\ ; \quad m = 1,...,13.$$

Thus in place of using the more appropriate annual average prices p_n^b defined by (101), the ONS approximates these prices by taking the monthly elementary index for January of the previous year (which is the approximate midpoint of the base year), $p_n^{y-1,1}$. It is likely that the ONS approximation does not lead to any appreciable bias for most years[101] but if it is possible to construct *monthly* expenditure share estimates for the base year, then the ONS could calculate the annual average base year prices p_n^b using definition (101) and then these better estimates for the base year prices could be used in place of the approximate base year prices $p_n^{y-1,1}$ in their price updating procedures.

5.3 Two Stage Aggregation and Lowe Indexes

Another practical problem which was not considered in any detail up to now is the fact that the Lowe indexes that the ONS calculates do not use the single stage of aggregation methodology which was used in section 5.1. The prices p_n^b and $p_n^{y,m}$ which appear in the various formulae for the Lowe index in section 5.1 are actually elementary indexes for the N strata under consideration. Under what conditions will these various formulae for the Lowe index be equal to a true Lowe index? This is the question which we will now address.

The problems associated with reconciling two stage aggregation with single stage aggregation can be illustrated if the overall index consists of only two strata. Some new notation needs to be introduced. Let p_{nk}^b, q_{nk}^b and e_{nk}^b denote the base year prices, quantities and expenditures for the kth item in stratum n where n = 1,2 and k = 1,2,...,K(n). Using the same commodity classification, let $p_{nk}^{y,m}$, $q_{nk}^{y,m}$ and $e_{nk}^{y,m}$ denote the (augmented) year y and month m prices, quantities and expenditures for the kth item in stratum n where n = 1,2; k = 1,2,...,K(n) and m = 1,2,...,13. Thus there are K(1) separate items in the first stratum and K(2) items in the second stratum. Then the *true Lowe index for month m in the augmented year y*, constructed in a single stage, is defined as follows:

$$(116)\ P_{Lo}(p_1^b,p_2^b,p_1^{y,1},p_2^{y,1},p_1^{y,m},p_2^{y,m},s_1^b.s_2^b,e_1^b,e_2^b) \quad m= 1,2,...,13$$

$$\equiv [\sum_{k=1}^{K(1)} p_{1k}^{y,m}q_{1k}^b + \sum_{k=1}^{K(2)} p_{2k}^{y,m}q_{2k}^b]/[\sum_{k=1}^{K(1)} p_{1k}^{y,1}q_{1k}^b + \sum_{k=1}^{K(2)} p_{2k}^{y,1}q_{2k}^b]$$

[101] The use of January prices in the base year in place of true average prices for the base year will lead to more volatility in the price updated weights for the current year since the prices for a single month will tend to be more volatile than the corresponding annual prices.

$$= \frac{\sum_{k=1}^{K(1)} (p_{1k}^{y,m} / p_{1k}^{y,1})(p_{1k}^{y,1} / p_{1k}^{b}) s_{1k}^{b} e_1^b + \sum_{k=1}^{K(2)} (p_{2k}^{y,m} / p_{2k}^{y,1})(p_{2k}^{y,1} / p_{2k}^{b}) s_{2k}^{y,1} e_2^b}{\sum_{k=1}^{K(1)} (p_{1k}^{y,1} / p_{1k}^{b}) s_{1k}^{b} e_1^b + \sum_{k=1}^{K(2)} (p_{2k}^{y,1} / p_{2k}^{b}) s_{2k}^{b} e_2^b}$$

where $p_1^b \equiv [p_{11}^b, p_{12}^b, ..., p_{1K(1)}^b]$ and $p_2^b \equiv [p_{21}^b, p_{22}^b, ..., p_{2K(2)}^b]$ are the base year price vectors for strata 1 and 2; $p_n^{y,m} \equiv [p_{n1}^{y,m}, p_{n2}^{y,m}, ..., p_{nK(n)}^{y,m}]$ is the augmented year y, month m vector of prices for stratum n for n = 1,2 and m = 1,2,...,13; $s_n^b \equiv [s_{n1}^b, s_{n2}^b, ..., s_{nK(n)}^b]$ is the vector of base year expenditure shares for stratum n for n = 1,2 and finally, e_1^b and e_2^b are total expenditures on strata 1 and 2 respectively in the base year. The second expression for the true Lowe index in equations (116) looks rather formidable but it will prove to be useful below.

Now suppose that the statistical agency has constructed *elementary indexes* for the two strata. For stratum n, suppose that the elementary index level for the base year is P_n^b for n = 1,2. The corresponding (approximate) *base year quantities* for the two strata, Q_n^b, are defined as follows:

(117) $Q_n^b \equiv e_n^b / P_n^b$; n = 1,2.

We further suppose that the elementary index levels for stratum n for month m in the augmented year y are defined by $P_n^{y,m}$ for n = 1,2 and m = 1,2,...,13.

The following *two stage (approximate) Lowe index*, P_{LoA}, constructed using the elementary indexes and the approximate base year quantities defined by (117), is defined as follows:

(118) $P_{LoA}(P^b, P^{y,1}, P^{y,m}, e^b) \equiv [P_1^{y,m} Q_1^b + P_2^{y,m} Q_2^b] / [P_1^{y,1} Q_1^b + P_2^{y,1} Q_2^b]$; m = 1,2,...,13
$= [(P_1^{y,m}/P_1^b) e_1^b + (P_2^{y,m}/P_2^b) e_2^b] / [(P_1^{y,1}/P_1^b) e_1^b + [(P_2^{y,1}/P_1^b) e_2^b]$ using (117)
$= [(P_1^{y,m}/P_1^{y,1})(P_1^{y,1}/P_1^b) e_1^b + (P_2^{y,m}/P_2^{y,1})(P_2^{y,1}/P_2^b) e_2^b] / [(P_1^{y,1}/P_1^b) e_1^b + [(P_2^{y,1}/P_1^b) e_2^b]$.

Under what conditions will the approximate Lowe index P_{LoA} defined by (118) be equal to the true Lowe index P_{Lo} defined by (116)? An intuitively appealing set of sufficient conditions for equality are:

- Within each stratum, prices move in a proportional manner and
- The elementary indexes capture these proportional movements in prices.

The price proportionality assumptions for each stratum can formally be represented by the following equations; there exist positive constants α_n^m such that prices within the augmented year y satisfy the following equations:[102]

(119) $p_{nk}^{y,m} = \alpha_n^m p_{nk}^{y,1}$; n = 1,2 ; k = 1,2,...,K(n) ; m = 1,2,...,13.

[102] When m = 1, equations (119) are automatically satisfied.

We will also require that prices within a stratum move in a proportional manner going from the base year b to the first month of year y; i.e., there exist positive constants β_n such that the month 1, year y price vector for stratum n, $p_n^{y,1}$ is proportional to the corresponding base year stratum price vector p_n^b; i.e., we have the existence of positive constants β_1 and β_2 such that the following equations are satisfied:

(120) $p_{nk}^{1,m} = \beta_n p_{nk}^b$; n = 1,2 ; k = 1,2,...,K(n).

Finally, the assumption that the elementary indexes capture the same trends in prices that are present in each stratum can be represented algebraically by the following assumptions:

(121) $P_n^{y,m}/P_n^{y,1} = \alpha_n^m$; n = 1,2 ; m = 1,2,...,13;

(122) $P_n^{y,1}/P_n^b = \beta_n$; n = 1,2.

If we substitute assumptions (121) and (122) into the approximate Lowe index defined by (118), we find that:

(123) $P_{LoA} = [\alpha_1^m \beta_1 e_1^b + \alpha_2^m \beta_2 e_2^b]/[\beta_1 e_1^b + \beta_2 e_2^b]$; m = 1,2,,,,13.

If we substitute assumptions (119) and (120) into the true Lowe index defined by (116), we find that

(124) $P_{Lo}(p_1^b,p_2^b,p_1^{y,1},p_2^{y,1},p_1^{y,m},p_2^{y,m},s_1^b.s_2^b,e_1^b,e_2^b)$ m= 1,2,...,13

$= [\sum_{k=1}^{K(1)} \alpha_1^m\beta_1 s_{1k}^b e_1^b + \sum_{k=1}^{K(2)} \alpha_2^m\beta_2 s_{2k}^b e_2^b]/[\sum_{k=1}^{K(1)} \beta_1 s_{1k}^b e_1^b + \sum_{k=1}^{K(2)} \beta_2 s_{2k}^b e_2^b]$

$= [\alpha_1^m \beta_1 e_1^b + \alpha_2^m \beta_2 e_2^b]/[\beta_1 e_1^b + \beta_2 e_2^b]$ since $\sum_{k=1}^{K(n)} s_{nk}^b = 1$ for each n

$= P_{LoA}(P^b,P^{y,1},P^{y,m},e^b)$ using (123).

Thus if prices within each stratum vary proportionally over time and the elementary indexes capture these proportional movements in prices, then the approximate Lowe index that is constructed in two stages using the elementary indexes in the first stage will be equal to the true Lowe index. While the assumptions underlying this result are not likely to hold in practice, they may be approximately true and so the Lowe type indexes constructed by the ONS will approximate true Lowe indexes under these conditions.

5.4 Problems Associated with the Use of Annual Baskets in a Monthly Index

We conclude our discussion of Lowe indexes with some problematic aspects of Lowe indexes that use annual baskets in the context of producing monthly price indexes.

It should be noted that the problems associated with the Lowe index that uses an annual basket in the context of a monthly price index have been noted in the literature on price indexes when there are seasonal commodities. In the context of seasonal price indexes, the Lowe index is known as the Bean and Stine (1924; 31) Type A index or an Annual

Basket (AB) index. The price statistician Andrew Baldwin's comments on this type of index are worth quoting at length:[103]

"For seasonal goods, the AB index is best considered an index partially adjusted for seasonal variation. It is based on annual quantities, which do not reflect the seasonal fluctuations in the volume of purchases, and on raw monthly prices, which do incorporate seasonal price fluctuations. Zarnowitz (1961; 256-257) calls it an index of 'a hybrid sort'. Being neither of sea nor land, it does not provide an appropriate measure either of monthly or 12 month price change. The question that an AB index answers with respect to price change from January to February say, or January of one year to January of the next, is 'What would have the change in consumer prices have been if there were no seasonality in purchases in the months in question, but prices nonetheless retained their own seasonal behaviour?' It is hard to believe that this is a question that anyone would be interested in asking." Andrew Baldwin (1990; 258).

Basically, the problem is that households do not purchase (fractions) of their annual basket of purchases for each month of the given year; i.e., there is a *seasonal pattern* to their purchases. Thus for each month of the year, there will be an appropriate *monthly basket* that is relevant for index number construction rather than an annual basket. The problem of seasonal commodities in an annual basket index becomes apparent for *strongly seasonal commodities*; these are commodities that are present in some months of the year but not all months.

Are there solutions to the index number problems generated by seasonal commodities? If not, the Lowe index with some suitable modifications may still be the best index that can be produced under the circumstances. Thus in the following section, we will review where the current state of theory is with respect to producing monthly price indexes when seasonality is present.

[103] Balk (1980; 68c) also clearly pointed out the problems associated with using annual weights and monthly prices in the context of seasonal commodities.

6. The Problem of Seasonal Commodities

6.1 Introduction

Most of the material on the treatment of seasonal commodities that will be presented in this section is in Chapter 22 of the *Consumer Price Index Manual*.

It should be noted that all of the methods that will be suggested to deal with seasonal commodities assume that the statistical agency is able to collect expenditure information on these seasonal commodities by month. This expenditure information may be collected on a delayed basis through household expenditure surveys or possibly by obtaining detailed price and quantity data from retailers on a current basis (as is the case in the Netherlands).

Seasonal commodities are commodities which are either: (a) not available in the marketplace during certain seasons of the year or (b) are available throughout the year but there are regular fluctuations in prices or quantities that are synchronized with the season or the time of the year.[104] A commodity that satisfies (a) is termed a *strongly seasonal commodity* whereas a commodity which satisfies (b) will be called a *weakly seasonal commodity*.[105]

Strongly seasonal commodities offer the biggest challenge to traditional bilateral index number theory, which assumes that all commodities are present in both periods being compared. Obviously, it is not possible to compare the price of a strongly seasonal commodity in a month where it is present in the marketplace with a nonexistent price for the commodity in a month where it is not available at all. However, even weakly seasonal commodities offer challenges to traditional index number theory since an increase in the index in a given month may simply be due to a seasonal increase in prices rather than a "true" increase in underlying inflation for the reference population.

In order to deal with seasonal commodities, the *Consumer Price Index Manual* offers *four types of index*:

- Year over year monthly indexes;
- Year over year annual indexes;
- Rolling year annual indexes and
- Month to month (chained) indexes.

In the following four subsections, each type of index will be defined and (briefly) discussed. Section 6.6 will conclude that the first three types of index are conceptually sound but the fourth type of index suffers from a chain drift problem and hence is not a

[104] This classification of seasonal commodities corresponds to Balk's narrow and wide sense seasonal commodities; see Balk (1980a; 7) (1980b; 110) (1980c; 68). Diewert (1998; 457) used the terms type 1 and type 2 seasonality.

[105] Mitchell (1927) noted that seasonal variations in prices and quantities are basically due to (i) fluctuations in the weather and (ii) customs.

suitable index for the ONS to consider producing. Section 7 below will offer an alternative month to month index that has superior properties.

Before proceeding to the technical definitions of the various indexes, it is necessary to discuss the notation that will be used and the interpretation of the variables. The algebra below will assume that the statistical agency has information on the monthly prices and quantities for the N commodities that enter the scope of the index. However, not all commodities will be present in each month. Denote the set of commodities n which are present in the marketplace during month m of any year as S(m).[106] Denote the price of commodity n in month m of year y as $p_n^{y,m}$ the corresponding quantity and expenditure share as $q_n^{y,m}$ and $s_n^{y,m} \equiv p_n^{y,m}q_n^{y,m}/\sum_{k\in S(m)} p_k^{y,m}q_k^{y,m}$ for y = 0,1, m = 1,2,...,12 and $n\in S(m)$.[107]

In the following four sections, various index number formulae will be defined using the above notation. However, the resulting indexes could refer to several situations:

- N is the total number of separate items that are to be distinguished in the overall consumer price index; i.e., the underlying assumption here is that we have complete price and quantity information on the universe of expenditures for the reference population.
- N refers to the number of items in one particular *stratum* of the overall consumer price index. Standard index number theory is also applicable in this situation.
- N refers to the number of strata in the consumer price index and in this case, the prices $p_n^{y,m}$ are in fact, elementary indexes, and the corresponding $s_n^{y.m}$ are expenditure shares on the nth elementary category or stratum.[108]

Obviously, application of the first interpretation of the indexes is unrealistic; the statistical agency will typically not have access to true microeconomic data at the finest level of aggregation. However, application of the second interpretation of the indexes is quite possible; the existence of scanner data sets has led to the possibility of computing

[106] We are making the simplifying assumption that the set of strongly seasonal commodities remains the same as the years change. In practice, this assumption does not always hold; see Diewert, Finkel and Artsev (2009; 63). In order to deal with this situation where a price is present in one year for a month but not in the same month for the other year, a *price* for the commodity should be imputed for the month when the commodity is not available. Basically, the imputed price should be based on price movements of products that are thought to be comparable to the given seasonal product; see Diewert, Finkel and Artsev (2009; 61) for one such imputation method. For more systematic discussions of imputation methods, see Alterman, Diewert and Feenstra (1999) and Diewert and Feenstra (2001). The corresponding imputed quantity and monthly expenditure share for the missing price should be set equal to zero.

[107] The summation $\sum_{k\in S(m)} p_k^{y,m}q_k^{y,m}$ means that we sum expenditures in month m of year y over products k that are actually present in month m; i.e., strongly seasonal products that are not present in month m are excluded in this sum.

[108] In this case, the corresponding quantities are defined implicitly as $q_n^{y,m} \equiv s_n^{y,m}/p_n^{y,m}$ for $n\in S(m)$. A useful assumption that can ensure that the indexes constructed under this framework are true indexes (e.g., true Fisher indexes that are based on microeconomic data at the finest level of aggregation) is that within each stratum, prices of the products within the stratum vary proportionally over time as in section 5.3 above.

say true Fisher indexes for some strata of the CPI.[109] The third interpretation of the indexes is of course directly relevant to the ONS. In what follows, the discussion of the indexes will use the first interpretation but the reader should keep in mind the more useful third interpretation.

6.2 Year over Year Monthly Indexes

For over a century,[110] it has been recognized that making year over year comparisons[111] of prices in the same month provides the simplest method for making comparisons that are free from the contaminating effects of seasonal fluctuations.

We will take the Fisher index as the "best" functional form for making bilateral comparisons. In the present context, the 12 *month over month Laspeyres*, *Paasche* and *Fisher* indexes, P_L, P_P and P_F, comparing the prices in year 1 for month m to those in year 0 for month m are defined as follows, using the notation in the previous subsection:

(125) $P_L(p^{0,m},p^{1,m},s^{0,m}) \equiv \sum_{n\in S(m)} (p_n^{1,m}/p_n^{0,m})s_n^{0,m}$; m= 1,...,12;
(126) $P_P(p^{0,m},p^{1,m},s^{1,m}) \equiv [\sum_{n\in S(m)} (p_n^{1,m}/p_n^{0,m})^{-1}s_n^{0,m}]^{-1}$; m= 1,...,12;
(127) $P_F(p^{0,m},p^{1,m}, s^{0,m},s^{1,m}) \equiv [P_L(p^{0,m},p^{1,m},s^{0,m})P_P(p^{0,m},p^{1,m},s^{1,m})]^{1/2}$; m = 1,...,12

where $p^{y,m}$ is the vector of prices for commodities that are present in month m of year y and $s^{y,m}$ is the corresponding expenditure vector for y = 0,1 and m = 1,...,12. In the *Consumer Price Index Manual*,[112] it was noted that approximate month over month Laspeyres, Paasche and Fisher indexes could be defined, using information that is generally available to the statistical agency, by replacing the monthly expenditure shares $s_n^{0,m}$ and $s_n^{1,m}$ by the available monthly expenditure shares for the current base year $s_n^{b,m}$ for m = 1,...,12. Thus the 12 *approximate month over month Laspeyres*, *Paasche* and *Fisher* indexes, P_{AL}, P_{AP} and P_{AF}, are defined as follows:

(128) $P_{AL}(p^{0,m},p^{1,m},s^{b,m}) \equiv \sum_{n\in S(m)} (p_n^{1,m}/p_n^{0,m})s_n^{b,m}$; m= 1,...,12;
(129) $P_{AP}(p^{0,m},p^{1,m},s^{b,m}) \equiv [\sum_{n\in S(m)} (p_n^{1,m}/p_n^{0,m})^{-1}s_n^{b,m}]^{-1}$; m= 1,...,12;
(130) $P_{AF}(p^{0,m},p^{1,m},s^{b,m}) \equiv [P_L(p^{0,m},p^{1,m},s^{b,m})P_P(p^{0,m},p^{1,m},s^{b,m})]^{1/2}$; m = 1,...,12

where $s^{b,m}$ is the vector expenditure shares for month m in the base year b.[113] The approximate Fisher year over year monthly indexes defined by (130) will provide adequate approximations to their true Fisher counterparts defined by (127) if the monthly expenditure shares for the base year b are not too different from their current year 0 and 1 counterparts. Hence, it will be useful to construct the true Fisher indexes on a delayed

[109] See Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011) for applications of this type. We will discuss what can be done when scanner data are available in more detail in section 7 below.
[110] See Jevons (1884;3), Flux (1921; 199) and Yule (1921; 199).
[111] In the seasonal price index context, this type of index corresponds to Bean and Stine's (1924; 31) Type D index.
[112] See the ILO (2004; 397).
[113] Thus this vector excludes the strongly seasonal commodities that are not present in month m.

basis in order to check the adequacy of the approximate Fisher indexes defined by (130).[114]

The year over year monthly approximate Fisher indexes defined by (130) will normally have a certain amount of upward bias, since these indexes cannot reflect long term substitution of consumers towards commodities that are became relatively cheaper over time. This reinforces the case for computing true year over year monthly Fisher indexes defined by (127) on a delayed basis so that this substitution bias can be estimated.

In the following section, it is shown how the various year over year monthly indexes defined in this section can be aggregated into an annual index.

6.3 Annual Year over Year Indexes

Mudgett (1955) in the consumer price context and Stone (1956) in the producer price context independently suggested a very effective way of dealing with seasonal commodities in the context of constructing an annual index:

"The basic index is a yearly index and as a price or quantity index is of the same sort as those about which books and pamphlets have been written in quantity over the years." Bruce D. Mudgett (1955; 97).

"The existence of a regular seasonal pattern in prices which more or less repeats itself year after year suggests very strongly that the varieties of a commodity available at different seasons cannot be transformed into one another without cost and that, accordingly, in all cases where seasonal variations in price are significant, the varieties available at different times of the year should be treated, in principle, as separate commodities." Richard Stone (1956; 74-75).

Thus the basic idea is that each commodity in each month should be treated as a distinct commodity in an annual index. Thus the price of commodity n in January of year 1 is compared to the price of the same commodity in January of year 0 and so on.

Using the notation introduced in the previous section, the *Laspeyres, Paasche and Fisher annual indexes* comparing the prices of year 0 with those of year 1 can be defined as follows:

$$(131)\quad P_L(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};q^{0,1},...,q^{0,12}) \equiv \sum_{m=1}^{12} \sum_{n\in S(m)} p_n^{1,m}q_n^{0,m} / \sum_{m=1}^{M} \sum_{n\in S(m)} p_n^{0,m}q_n^{0,m} ;$$

$$(132)\quad P_P(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};q^{1,1},...,q^{1,12}) \equiv \sum_{m=1}^{12} \sum_{n\in S(m)} p_n^{1,m}q_n^{1,m} / \sum_{m=1}^{M} \sum_{n\in S(m)} p_n^{0,m}q_n^{1,m} ;$$

$$(133)\quad P_F(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};q^{0,1},...,q^{0,12};q^{1,1},...,q^{1,12}) \equiv$$

[114] For the modified Turvey (1979) data set in the *Consumer Price Index Manual*, the approximate Fisher indexes were quite close to the true Fisher indexes; see the ILO (2004; 398-400). However, for the Israeli data set on fresh fruits, the correspondence between the true and approximate Fishers was not close for a large number of observations; i.e., for 16 out of 60 observations, the difference exceeded 5%: see Diewert, Finkel and Artsev (2009; 59-60). The Israeli example shows the importance of computing the true Fishers on a delayed basis when the true expenditure information becomes available.

$$[P_L(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};q^{0,1},...,q^{0,12})P_P(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};q^{1,1},...,q^{1,12})]^{1/2}.$$

The above formulae can be rewritten in terms of price relatives and monthly expenditure shares as follows:

$$(134)\quad P_L(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};\sigma^{0,1}s^{0,1},...,\sigma^{0,12}s^{0,12}) \equiv \sum_{m=1}^{12} \sum_{n\in S(m)} \sigma^{0,m}s_n^{0,m} (p_n^{1,m}/p_n^{0,m}) = \sum_{m=1}^{12} \sigma^{0,m} P_L(p^{0,m},p^{1,m},s^{0,m})\ ;$$

$$(135)\quad P_P(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};\sigma^{1,1}s^{1,1},...,\sigma^{1,12}s^{1,12}) \equiv [\sum_{m=1}^{12} \sum_{n\in S(m)} \sigma^{1,m}s_n^{1,m} (p_n^{1,m}/p_n^{0,m})^{-1}]^{-1} = [\sum_{m=1}^{12} \sigma^{1,m} \sum_{n\in S(m)} s_n^{1,m} (p_n^{1,m}/p_n^{0,m})^{-1}]^{-1} = [\sum_{m=1}^{12} \sigma^{1,m} [P_P(p^{0,m},p^{1,m},s^{1,m})]^{-1}]^{-1}\ ;$$

$$(136)\quad P_F(p^{0,1},...,p^{0,12};p^{1,1},...,p^{1,12};\sigma^{0,1}s^{0,1},...,\sigma^{0,12}s^{0,12};\sigma^{1,1}s^{1,1},...,\sigma^{1.12}s^{1,12}) \equiv [\sum_{m=1}^{12} \sum_{n\in S(m)} \sigma^{0,m}s_n^{0,m} (p_n^{1,m}/p_n^{0,m})\{\sum_{m=1}^{12} \sum_{n\in S(m)} \sigma^{1,m}s_n^{1,m} (p_n^{1,m}/p_n^{0,m})^{-1}\}^{-1}\}]^{1/2} = [\{\sum_{m=1}^{12} \sigma^{0,m} P_L(p^{0,m},p^{1,m},s^{0,m})\}\{\sum_{m=1}^{12} \sigma^{1.m} [P_P(p^{0,m},p^{1,m},s^{1,m})]^{-1}\}^{-1}]^{1/2}$$

where *the expenditure share for month m in year t* is defined as:

$$(137)\quad \sigma^{t,m} \equiv \sum_{n\in S(m)}\ p_n^{t,m}q_n^{t,m} / \sum_{i=1}^{12} \sum_{j\in S(i)} p_j^{t,i}q_j^{t,i}\ ; \qquad m = 1,2,...,12\ ;\ t = 0,1$$
$$= p^{t,m}\cdot q_n^{t,m} / \sum_{i=1}^{12} p^{t,i}\cdot q^{t,i}$$

and the year over year monthly Laspeyres and Paasche price indexes $P_L(p^{0,m},p^{1,m},s^{0,m})$ and $P_P(p^{0,m},p^{1,m},s^{1,m})$ were defined by (125) and (126) respectively. As usual, the annual Fisher index P_F defined by (136) is the geometric mean of the Laspeyres and Paasche indexes, P_L and P_P, defined by (134) and (135). The last equations in (134) and (135) shows that the annual Laspeyres index can be written as a *weighted arithmetic average* of the year over year monthly Laspeyres indexes $P_L(p^{0,m},p^{1,m},s^{0,m})$, where the weights are the monthly expenditure shares $\sigma^{0,m}$ in year 0, and the annual Paasche index can be written as a *weighted harmonic average* of the year over year monthly Paasche indexes $P_P(p^{0,m},p^{1,m},s^{1,m})$, where the weights are the monthly expenditure shares $\sigma^{1,m}$ in year 1. Hence once the year over year monthly indexes defined in the previous subsection have been numerically calculated, it is easy to calculate the corresponding annual indexes.

The approach to computing annual indexes outlined in this subsection, which essentially involves taking monthly expenditure share weighted averages of the 12 year over year monthly indexes, should be contrasted with the usual approach to the construction of annual indexes that simply takes the arithmetic mean of the 12 monthly indexes. The problem with the latter approach is that months where expenditures are below the average (e.g., February) are given the same weight in the unweighted annual average as months where expenditures are above the average.[115]

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[115] In addition, the Mudgett-Stone approach to the construction of annual indexes can be given a strong justification from the viewpoint of the economic approach to index number theory.

As in section 6.2, *Approximate Laspeyres, Paasche and Fisher counterparts* to the true indexes defined by (134)-(136) can be defined by these same equations, except that the monthly expenditure shares, $\sigma^{0,m}$ and $\sigma^{1,m}$, are replaced by the corresponding base year expenditure shares, $\sigma^{b,m}$, for m = 1,...,12 and the within the month expenditure shares, $s_n^{0,m}$ and $s_n^{1,m}$, are replaced by the corresponding within the month base year expenditure shares, $s_n^{b,m}$ for m = 1,...,12 and $n \in S(m)$.

For the artificial data set in the *Consumer Price Index Manual*, the approximate annual Fisher index provided a very close approximation to the true Fisher indexes. This result needs further validation but it should be noted that the approximate Fisher index can be computed using the same information set that is normally available to statistical agencies.

6.4 Rolling Year Annual Indexes

In the previous subsection, the price and quantity data pertaining to the 12 months of a calendar year were compared to the 12 months of a base calendar year. However, there is no need to restrict attention to calendar year comparisons: any 12 consecutive months of price and quantity data could be compared to the price and quantity data of the base year, provided that the January data in the noncalendar year is compared to the January data of the base year, the February data of the noncalendar year is compared to the February data of the base year, …, and the December data of the noncalendar year is compared to the December data of the base year.[116] Alterman, Diewert and Feenstra (1999; 70) called the resulting indexes *rolling year* or *moving year* indexes.[117]

In order to theoretically justify the rolling year indexes from the viewpoint of the economic approach to index number theory, some restrictions on preferences are required. The details of these assumptions can be found in Diewert (1999a; 56-61).

Of course, approximate counterpart to rolling year Laspeyres, Paasche and Fisher indexes can be obtained by replacing the various expenditure shares in the current year by their counterparts in the base year. The details and an example based on the Turvey data can be found in pages 402-406 of the *Consumer Price Index Manual*. The bottom line is that the true rolling year indexes were very smooth and free of seasonal influences and the approximate rolling year Fisher index approximated its true counterpart very well.[118]

Basically, the rolling year indexes are generally quite smooth and free from seasonal fluctuations. Each rolling year index can be viewed as a *seasonally adjusted annual consumer price index* that compares the data of the 12 consecutive months that end with the current month with the corresponding price and quantity data of the 12 months in the base year. Thus rolling year indexes offer statistical agencies an *objective* and

[116] Diewert (1983) suggested this type of comparison and termed the resulting index a "split year" comparison.
[117] Crump (1924; 185) and Mendershausen (1937; 245) respectively used these terms in the context of various seasonal adjustment procedures. The term "rolling year" seems to be well established in the business literature in the UK.
[118] Similar results were obtained using the Israeli data; see Diewert, Artsev and Finkel (2009; 61).

reproducible method of seasonal adjustment that can compete with existing time series methods of seasonal adjustment.[119] Rolling year indexes are probably the most reliable indexes that can be constructed when there is substantial seasonality in prices.

There are two main difficulties with the use of rolling year indexes:

- The measure of inflation generated by the current index value is for a rolling year that is centered around the month that took place six months ago and
- It is not a valid measure of month to month inflation.

Thus in the following subsection, month to month inflation indexes are considered.

6.5 Maximum Overlap Month to Month Price Indexes

A possible method for dealing with seasonal commodities in the context of picking a target index for a month to month CPI is the following one:[120]

- Determine the set of commodities that are present in the marketplace in both months of the comparison.
- For this maximum overlap set of commodities, calculate one of the three indexes recommended in previous sections; i.e., calculate the Fisher, Walsh or Törnqvist Theil index.

Thus the bilateral index number formula is applied only to the subset of commodities that are present in both periods.[121]

The question now arises: should the comparison month and the base month be adjacent months (thus leading to chained indexes) or should the base month be fixed (leading to fixed base indexes)? In the *Consumer Price Index Manual*, a preference was expressed for chained indexes over fixed base indexes for the following two reasons:[122]

- The set of seasonal commodities which are present in the marketplace during two consecutive months is likely to be much larger than the set obtained by comparing the

[119] For discussions on the merits of econometric or time series methods versus index number methods of seasonal adjustment, see Diewert (1999a; 61-68) and Alterman, Diewert and Feenstra (1999; 78-110). The basic problem with time series methods of seasonal adjustment is that the target seasonally adjusted index is very difficult to specify in an unambiguous way; i.e., there are an infinite number of possible target indexes. For example, it is impossible to identify a temporary increase in inflation within a year from a changing seasonal factor. Hence different econometricians will tend to generate different seasonally adjusted series, leading to a lack of reproducibility.

[120] For more on the economic approach and the assumptions on consumer preferences that can justify month to month maximum overlap indexes, see Diewert (1999a; 51-56).

[121] Keynes (1930; 95) called this the highest common factor method for making bilateral index number comparisons. Of course, this target index drops those strongly seasonal commodities that are not present in the marketplace during one of the two months being compared. Thus the index number comparison is not completely comprehensive. Mudgett (1951; 46) called the "error" in an index number comparison that is introduced by the highest common factor method (or maximum overlap method) the "homogeneity error".

[122] See the ILO (2004; 407-411).

prices of any given month with a fixed base month (like January of a base year). Hence the comparisons made using chained indexes will be more comprehensive and hence more accurate than those made using a fixed base.

- In many economies, on average 2 or 3 percent of price quotes disappear each month due to the introduction of new commodities and the disappearance of older ones. This rapid sample attrition means that fixed base indexes rapidly become unrepresentative and hence it seems preferable to use chained indexes which can more closely follow marketplace developments.

Thus in the present subsection, we will follow the advice in the Manual and define the relevant Laspeyres, Paasche and Fisher maximum overlap month to month chain links. Some new notation is required. Let there be N commodities that are available in at least one month of the current year and let $p_n^{y,m}$ and $q_n^{y,m}$ denote the price and quantity of commodity n that is in the marketplace in month m of year y (if the commodity is unavailable, define $p_n^{y,m}$ and $q_n^{y,m}$ to be 0). Let $p^{y,m} \equiv [p_1^{y,m},p_2^{y,m},...,p_N^{y,m}]$ and $q^{y,m} \equiv [q_1^{y,m},q_2^{y,m},...,q_N^{y,m}]$ be the month m and year y price and quantity vectors respectively. Let S(y,m) be the *set of commodities that is present in month m of year y and the following month*. Then the *maximum overlap Laspeyres, Paasche and Fisher indexes* going from month m of year t to the following month can be defined as follows:[123]

$$(138)\ P_L(p^{y,m},p^{y,m+1},q^{t,m},S(y,m)) \equiv \sum_{n\in S(y,m)} p_n^{y,m+1}q_n^{y,m}/\sum_{n\in S(y,m)} p_n^{y,m}q_n^{y,m};\ m = 1,2,...11;$$

$$(139)\ P_P(p^{y,m},p^{y,m+1},q^{y,m+1},S(y,m)) \equiv \sum_{n\in S(y,m)} p_n^{y,m+1}q_n^{y,m+1}/\sum_{n\in S(y,m)} p_n^{y,m}q_n^{y,m+1};\quad m = 1,2,...11;$$

$$(140)\ P_F(p^{y,m},p^{y,m+1},q^{y,m},q^{y,m+1},S(y,m)) \equiv [P_L(p^{y,m},p^{y,m+1},q^{y,m},S(y,m))P_P(p^{y,m},p^{y,m+1},q^{y,m+1},S(y,m))]^{1/2};\quad m = 1,2,...,11.$$

Note that P_L, P_P and P_F depend on the two (complete) price and quantity vectors pertaining to months m and m+1 of year y, $p^{y,m},p^{y,m+1},q^{y,m},q^{y,m+1}$, but they also depend on the set S(y,m), which is the set of commodities that are present in both months. Thus the commodity indices n that are in the summations on the right hand sides of (138) and (139) include indices n that correspond to commodities that are present in *both* months, which is the meaning of $n\in S(y,m)$; i.e., n belongs to the set S(y,m).

We will not convert formulae (138) and (139) into expenditure share form; the details on how this can be done are in the *Consumer Price Index Manual*.[124]

The performance of maximum overlap chained indexes has not been satisfactory; indexes based on this methodology tend to suffer from a *chain drift problem*; i.e., indexes constructed using this methodology tend to exhibit a downward bias.[125]

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[123] The formulae are slightly different for the indices that go from December to January of the following year. In order to simplify the exposition, these formulae are left for the reader.

[124] See the ILO (2004; 408).

[125] For documentation of this phenomenon, see the ILO (2004; 409), Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011).

An objective method to test for the existence of chain drift in a price index $P(p^1,p^2,q^1,q^2)$ is the following *multiperiod identity test*,[126] which was initially proposed by Walsh (1901, 401):

$$(141)\ P(p^1,p^2,q^1,q^2)\ P(p^2,p^3,q^2,q^3)\ P(p^3,p^1,q^3,q^1) = 1.$$

$P(p^1,p^2,q^1,q^2)$ and $P(p^2,p^3,q^2,q^3)$ are price indexes between periods 1 and 2, and then 2 and 3, respectively, where p^t and q^t are the price and quantity vectors pertaining to periods t for t = 1, 2, 3. Their product gives the chained price index between periods 1 and 3. Note that there is a final link in the chain in (141), $P(p^3,p^1,q^3,q^1)$, which is a price index between periods 3 and 4, where the period 4 price and quantity data are the same as the period 1 data. The price index formula P will not suffer from *chain drift* or *chain link bias* if the product of all of these factors equals 1.

To see the relevance of the test defined by (141), suppose that in year y, month 1 and month 4 have exactly the same price and quantity data so that $p^{y,1} = p^{y,4}$ and $q^{y,1} = q^{y,4}$. Under these conditions, we would like the consumer price index chain links defined by (138)-(140) to be such that the month 1 and month 4 index values to be identical. Thus if the Fisher index is being used, in order to achieve this identity of index values for months 1 and 4 in year y, we require the following equality:

$$(142)\ P_F(p^{y,1},p^{y,2},q^{y,1},q^{y,2},S(y,1))P_F(p^{y,2},p^{y,3},q^{y,2},q^{y,3},S(y,2))\ P_F(p^{y,3},p^{y,1},q^{y,3},q^{y,1},S(y,3)) = 1.$$

The equality in (142) will not hold in general and thus chained maximum overlap month to month indexes will be subject to chain drift.[127]

What causes the downward drift of chained superlative indexes? A large part of the downward drift can be attributed to the problem of sales. A small numerical example will illustrate the problem.

Suppose that we are given the following price and quantity data for 2 commodities for 4 periods:

Table 3: Price and Quantity Data for Two Commodities

Period t	p_1^t	p_2^t	q_1^t	q_2^t
1	1.0	1.0	10	100
2	0.5	1.0	5000	100
3	1.0	1.0	1	100
4	1.0	1.0	10	100.

[126] Diewert (1993; 40) gave the test this name. Our exposition of how to define chain drift follows that of Ivancic, Diewert and Fox (2011; 26).

[127] The term "chain drift" seems to be due to Frisch: "The divergency which exists between a chain index and the corresponding direct index (when the latter does not satisfy the circular test) will often take the form of a systematic drifting." Ragnar Frisch (1936; 8).

The first commodity is subject to periodic sales (in period 2), when the price drops to one half of its normal level of 1. In period 1, the “normal” off sale demand for commodity 1 is equal to 10 units. In period 2, the sale takes place and demand explodes to 5000 units. In period 3, the commodity is off sale and the price is back to 1 but most shoppers have stocked up in the previous period so demand falls to only 1 unit. Finally in period 4, the commodity is off sale but we are back to the “normal” demand of 10 units. Commodity 2 is an unexciting commodity: its price is 1 in all periods and the quantity sold is 100 units in each period. Note that the only thing that has happened going from period 3 to 4 is that the demand for commodity one has picked up from 1 unit to 10 units. Also note that the period 4 data are exactly equal to the period 1 data so if Walsh’s test is satisfied, the product of the period to period chain links should equal one.

In Table 4, the fixed base Fisher, Laspeyres and Paasche price indexes, P_F, P_L and P_P and as expected, they behave well in period 4, returning to the period 1 level of 1. The chained Fisher, Törnqvist, Laspeyres and Paasche price indexes, P_{FCH}, P_{TCH}, P_{LCH} and P_{PCH} are listed. Obviously, the chained Laspeyres and Paasche indexes have chain link bias that is very large but what is interesting is that the chained Fisher has a 2% downward bias and the chained Törnqvist has a close to 3% downward bias.

Table 4: Fixed Base and Chained Fisher, Törnqvist, Laspeyres and Paasche Indexes

Period	P_F	P_L	P_P	P_{FCH}	P_{TCH}	P_{LCH}	P_{PCH}
1	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
2	0.69759	0.95455	0.50980	0.69759	0.69437	0.95455	0.50980
3	1.00000	1.00000	1.00000	0.97944	0.97232	1.87238	0.51234
4	1.00000	1.00000	1.00000	0.97944	0.97232	1.87238	0.51234

If the above data were monthly, and they repeated themselves 3 times over the year, the overall chain link bias would build up to the 6 to 8% range for the two chained superlative indexes, P_{FCH}, P_{TCH}, which is fairly large.

The sales problem can be explained as follows: when commodity one comes off sale and goes back to its regular price in period 3, the corresponding quantity does not return to the level it had in period 1: the period 3 demand is only 1 unit whereas the period 1 demand for commodity 1 was 10 units. It is only in period 4 that demand for commodity one recovers to the period 1 level. However, since prices are the same in periods 3 and 4, all of the chain links show no change (even though quantities are changing) and this is what causes the difficulties. If demand for commodity one in period 3 had immediately recovered to its “normal” period 1 level of 10, then there would be no problem with chain drift.

The problem of chain drift does not always lead to a downward bias in chained superlative indexes. Feenstra and Shapiro (2003) provide an empirical example where the chain drift bias goes in the opposite direction and they explain why this result occurred:

"The fixed base Törnqvist does not equal the chained Törnqvist in general, and for our sample of weekly tuna data, we find that the differences between these two indexes are rather large: the chained Törnqvist has a pronounced upward bias for most regions of the United States. The reason for this is that periods of low price (i.e., sales) attract high purchases only when they are accompanied by advertising, and this tends to occur in the final weeks of a sale. Thus, the initial price decline, when the sale starts, does not receive as much weight in the cumulative index as the final price *increase* when the sale ends. The demand behavior that leads to this upward bias of the chained Törnqvist—with higher purchases at the end of a sale—means that consumers are very likely purchasing goods for inventory accumulation. The only theoretically correct index to use in this type of situation is a fixed base index, as demonstrated in section 5.3." Robert Feenstra and Matthew Shapiro (2003, 125).

Thus Feenstra and Shapiro suggested that a solution to the problems generated by big fluctuations in prices and quantities due to sales is to move from chained indexes to fixed base indexes. In the following section, this suggestion will be implemented.

7. Scanner Data and the Rolling Year GEKS Method for Constructing Indexes

7.1 The ONS and Scanner Data

In early 2012, the ONS commenced Eurostat funded research on the exploitation of scanner data for the purpose of producing multipurpose consumer price statistics. Scanner data is defined as data on sales of consumer goods obtained by 'scanning' the barcodes for individual products at electronic point of sales systems. The data can provide detailed information about quantities, characteristics and values of goods sold as well as their prices. ONS research will focus on three defined stages of obtaining sample scanner data, implementing the data and testing the use of the scanner data in the construction of price indices. An assessment will be made of the feasibility of implementing a regular scanner data feed(s) into future production processes.

One of the potential benefits of using scanner data compared to traditional price collection methods is the availability of more frequent expenditure (or weight) information, at more detailed levels than in household surveys typically. The availability of this information presents an opportunity to calculate superlative indices, indices which treat prices and quantities equally across periods rather than assuming quantities are fixed for a period of time. There are three well known superlative indices – Fisher, Törnqvist and Walsh. However, recent literature has highlighted that these indexes can suffer from chain drift, this is where a chained index 'does not return to unity when prices in the current period return to their levels in the base period' (ILO, 2004). A newer methodological approach to the production of month to month indexes is suggested that avoids the chain drift problem. This new approach will be described in the following sub sections.

7.2 The Multilateral GEKS Method Applied in the Time Series Context

The GEKS method of making index number comparisons between multiple time periods or countries is due to Gini (1924; 110) (1931; 12). It was derived in a different fashion by Eltetö and Köves (1964) and Szulc (1964) and thus the method is known as either the GEKS or EKS method for making multilateral comparisons.

Some new notation will be introduced in order to explain the method in the time series context. Denote the price and quantity data for the past thirteen months by the superscript $c = 1, 2, ..., 13$. Thus when c equals 13, the data refer to the current month and when $c = 1$, the data refer to the same month one year ago from the current month. We call these thirteen months the *current augmented year*. As usual, let there be N commodities that are available in at least one month of the current augmented year and let p_n^c and q_n^c denote the price and quantity of commodity n that is in the marketplace in month c of the current augmented year. If the commodity is unavailable in month c, define p_n^c and q_n^c to be 0.[128] Let $p^c \equiv [p_1^c,p_2^c,...,p_N^c]$ and $q^c \equiv [q_1^c,q_2^c,...,q_N^c]$ be the price and quantity vectors for month c in the current augmented year. Let S(i,j) be the *set of commodities that is*

[128] If quantity information for the current augmented year is not available but price and expenditure information, p_n^c and e_n^c, is available, then if $e_n^c > 0$, define $q_n^c \equiv e_n^c/p_n^c$ and if $e_n^c = 0$, define $q_n^c \equiv 0$.

present in months i and j of the current augmented year, for i, j = 1,2,...,13. Then the *maximum overlap Laspeyres, Paasche and Fisher indexes* that compare the prices in month j to month i in the current augmented year are defined as follows:

(143) $P_L(j/i) \equiv \sum_{n\in S(i,j)} p_n^j q_n^i / \sum_{n\in S(i,j)} p_n^i q_n^i$; i.j = 1,2,...,13;
(144) $P_P(j/i) \equiv \sum_{n\in S(i,j)} p_n^j q_n^j / \sum_{n\in S(i,j)} p_n^i q_n^j$; i.j = 1,2,...,13;
(145) $P_F(j/i) \equiv [P_L(j/i)P_P(j/i)]^{1/2}$; i.j = 1,2,...,13.

The Fisher indexes $P_F(j/i)$ will have the usual satisfactory axiomatic properties, including satisfying the time reversal test ($P_F(i/j) = 1/P_F(j/i)$) and the identity test ($P_F(i/i) = 1$).

Now choose month k as the base month. Using k as the base month and the Fisher indexes as our formula, a complete set of index numbers for the augmented year can be obtained as the following sequence of 13 fixed base numbers: $P_F(1/k)$, $P_F(2/k)$, ..., $P_F(13/k)$. In the international comparisons literature, this set of price indexes for a fixed k is called the set *Fisher star PPPs with country k* as the star.[129] The final set of GEKS indexes for the 13 months is simply geometric mean of all 13 of the specific month star parities; i.e., the final set of *GEKS indexes for the months in the augmented year* is any normalization of the following indexes:[130]

(146) $[\prod_{k=1}^{13} P_F(1/k)]^{1/13}$, $[\prod_{k=1}^{13} P_F(2/k)]^{1/13}$, ... , $[\prod_{k=1}^{13} P_F(13/k)]^{1/13}$.

The above GEKS indexes have a number of important properties:[131]

- They satisfy Walsh's multiperiod identity test so that if any two months in the augmented year have exactly the same price and quantity vectors, then the above index values will coincide for those two months; i.e., *the above indexes are free from chain drift*.
- The above indexes do not asymmetrically single out any single month to play the role of a base period; all possible base months contribute to the overall index values.[132]
- The above indexes make use of all possible bilateral matches of the price data between any two months in the augmented year.
- Strongly seasonal commodities make a contribution to the overall index values.

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[129] This terminology follows that of Kravis (1984). In our present context, the "countries" are now the 13 months in the current augmented year.
[130] Balk (1981; 74) derived the GEKS parities using this type of argument rather than the usual least squares derivation of the GEKS parities; see Balk (1996) and Diewert (1999b) for these alternative derivations.
[131] The basic idea of adapting a multilateral method to the time series context is due to Balk (1981) who set up a framework that is very similar to the one explained here (which follows Ivancic, Diewert and Fox (2011) more closely). Balk (1981) used an index number formula due to Vartia (1976a) (1976b) in place of maximum overlap bilateral Fisher indexes as his basic building blocks and he considered augmented years of varying length instead of a 13 month augmented year but the basic idea of adapting multilateral methods to the time series context is due to him.
[132] Thus the above GEKS procedure is an improvement over the suggestion of Feenstra and Shapiro (2003) who chose only a single base month.

The last property explains why the augmented year should include at least 13 consecutive months, so that strongly seasonal commodities can make a contribution to the overall index.

The major problem with the GEKS indexes defined by (146) is that the indexes change as the data for a new month becomes available. A headline CPI cannot be revised from month to month due to the fact that many contracts are indexed to a country's headline consumer price index. A solution to this no revisions problem will be described in the following subsection.

7.3 Rolling Year GEKS Indexes

Ivancic, Diewert and Fox (2011) dealt with the fact that the addition of a new month's data would cause the GEKS indexes for past periods to change in the following way. Their method adds the price and quantity data for the most recent month to the augmented year and drops the oldest month from the old augmented year in order to obtain a new augmented year. The GEKS indexes for the new augmented year are calculated in the usual way and the ratio of the index value for the last month in the new augmented year to the index value for the previous month in the new augmented year is used as an *update factor* for the value of the index for the last month in the previous augmented year. The resulting indexes are called *Rolling Year GEKS indexes*.

Numerical experiments with Australian and Dutch scanner data from grocery chains show that the Rolling Year GEKS indexes work well when up to date price and quantity data are made available to the statistical agency; see Ivancic, Diewert and Fox (2011) and de Haan and van der Grient (2011). In particular, adding and dropping a month of data and recomputing the GEKS indexes does not seem to change past index values very much.[133]

A more recent paper that describes the Dutch experience with Rolling Year GEKS is by van der Grient and de Haan (2011) and we will briefly describe the contents of this important paper. The authors note that in 2010, Statistics Netherlands expanded the use of scanner data in their CPI using the monthly unit value data by detailed product provided by six supermarket chains. The six chains for which scanner data were included in the January 2010 CPI have an aggregate market share of some 50% and a weight of slightly over 5% in the CPI. They also described the new method that utilized this scanner data in the Dutch CPI. Monthly chained Jevons price indexes are now computed at the lowest aggregation level. However, not all item prices are used: the authors noted that using all item prices in the various elementary indexes overstates the importance of low expenditure items. Thus in the Dutch CPI, an item price is used in the computation of the price index between two adjacent months if its average expenditure share in the current and preceding month with respect to the set of matched items is above a certain threshold value. The threshold (for including the item in the Jevons elementary

[133] Balk (1981; 77) also observed the same phenomenon as he computed his GEKS indexes using successively larger data sets.

aggregate) was chosen so that roughly 50 percent of the items are selected, representing 80 to 85 percent of aggregate expenditure. This cut-off sampling was done at the elementary aggregation level; i.e., for each product category at the most detailed level within each supermarket chain. Van der Grient and de Haan (2011) noted that the lack of weighting at the item level is an obvious weakness. Thus the Dutch Central Bureau of Statistics is also computing Rolling Year GEKS indexes to compare with their cut-off sampling method for constructing elementary aggregates, which are used in their official CPI. In order to make an informed decision about possible implementation of the Rolling Year GEKS method, Statistics Netherlands has a shadow system running which computes rolling year GEKS indexes for each COICOP category and each supermarket chain. Van der Grient and de Haan (2011) fully describe both the official Dutch cut-off sampling method and the Rolling Year GEKS method and they present monthly index numbers for 2009 and 2010. At the all items level, they find that the two methods yield similar results. However, as might be expected, at lower levels of aggregation, there appear to be some marked differences.[134]

The problems associated with the treatment of quality change have not been mentioned in this report but quality adjusting prices of products that are subject to rapid technological change is an important problem. One important method for the treatment of quality change is the *hedonic regression* technique. This method regresses the price of the product on its price determining characteristics. It is outside the scope of this report to discuss this method in detail but an important new paper by de Haan and Krsinich (2012) shows how hedonic regression techniques can be combined with the Rolling Year GEKS method in the scanner data context.

Our conclusion at this point is that Rolling Year GEKS appears to be the “best” method for constructing elementary aggregates, provided supermarket chains are willing to provide the statistical agency with the required data on prices and quantities.

7.4 Approximate Rolling Year GEKS Indexes

The methodology that is used in constructing Rolling Year GEKS indexes for some elementary aggregates can be adapted for use in situations where the statistical agency has only “traditional” elementary aggregate information (based on a sample of item prices) and base year expenditure information for a rolling year in the past.[135]

The notation that was explained in section 7.2 above will be used here, with some modifications. Denote the (elementary) price index data for the past thirteen months by the superscript $c = 1, 2, ..., 13$. Thus when c equals 13, the data refer to the current month and when $c = 1$, the data refer to the same month one year ago from the current month. As in section 7.2, we call these thirteen months the *current augmented year*. Suppose that there are N elementary aggregates and let p_n^c denote the elementary price index level for

[134] Other countries who are experimenting with using supermarket scanner data in their consumer price indexes are Norway and Australia.

[135] This base year expenditure information is required for each month in the base year in order to implement the method suggested here.

stratum n in month c of the current augmented year for n = 1,...,N. If the stratum n corresponds to strongly seasonal items and in month c, no items are available, then define p_n^c to be 0. It is assumed that quantity or expenditure information for the current augmented year is not available but expenditure information by month is available for each stratum for an augmented base year b. Reorder the augmented base year expenditure information so that the monthly expenditures by month in the augmented base year line up with the months in the current augmented year. Thus if month 13 in the current augmented year is July of the current year, e_n^{13*} corresponds to the July expenditures for elementary stratum n in the base year; e_n^{12*} corresponds to the base year expenditures for stratum n in June of the base year and so on. These augmented base year expenditures by stratum and month can be converted into implicit quantities by dividing the augmented base year expenditures by the corresponding elementary price indexes p_n^{c*}. Thus if $e_n^{c*} > 0$, define $q_n^{c*} \equiv e_n^{c*}/p_n^{c*}$ and if $e_n^{c*} = 0$, define $q_n^{c*} \equiv 0$. Define the vector of elementary price indexes for month c of the current augmented year as $p^c \equiv [p_1^c,p_2^c,...,p_N^c]$ and define the vector of quantities for month c in the augmented base year as $q^{c*} \equiv [q_1^{c*},q_2^{c*},...,q_N^{c*}]$ for c = 1,...,13. Let S(i,j) be the *set of elementary strata that have positive elementary indexes for months i and j of the current augmented year*, for i, j = 1,2,...,13. Then the *approximate maximum overlap Laspeyres, Paasche and Fisher indexes* that compare the prices in month j to month i in the current augmented year are defined as follows:

$$(147)\ P_{AL}(j/i) \equiv \sum_{n\in S(i,j)} p_n^j q_n^{i*} / \sum_{n\in S(i,j)} p_n^i q_n^{i*}; \qquad i.j = 1,2,...,13;$$
$$(148)\ P_{AP}(j/i) \equiv \sum_{n\in S(i,j)} p_n^j q_n^{j*} / \sum_{n\in S(i,j)} p_n^i q_n^{j*}; \qquad i.j = 1,2,...,13;$$
$$(149)\ P_{AF}(j/i) \equiv [P_{AL}(j/i)P_{AP}(j/i)]^{1/2}; \qquad i.j = 1,2,...,13.$$

The indexes defined by (147)-(149) are the approximate counterparts to the "true" Laspeyres, Paasche and Fisher indexes defined earlier by (143)-(145): basically, current month c implicit quantity vectors q^c are replaced by their base year counterparts, q^{c*}.

Once the above approximate indexes have been defined, we can follow the methodology explained in section 7.1 above. Thus define the set of *approximate GEKS indexes for the months in the augmented year* as any normalization of the following indexes:

$$(150)\ [\prod_{k=1}^{13} P_{AF}(1/k)]^{1/13}, [\prod_{k=1}^{13} P_{AF}(2/k)]^{1/13}, ... , [\prod_{k=1}^{13} P_{AF}(13/k)]^{1/13} .$$

The above approximate GEKS indexes have a number of important properties:

- They satisfy a modification of Walsh's multiperiod identity test so that if any two months in the augmented year have exactly the same price vectors and the implicit quantity vectors for those two months in the base augmented year are the same, then the above index values will coincide for those two months; i.e., *the above indexes are free from chain drift*.

- The above indexes do not asymmetrically single out any single month to play the role of a base period; all possible base months contribute to the overall index values.[136]
- Strongly seasonal commodities make a contribution to the overall index values.

Now follow the Rolling Year GEKS methodology explained in section 7.2 above. Thus we add the price data for the most recent month to the current augmented year and drop the price data for the oldest month in order to obtain a new augmented year for prices. Similarly, we add an additional month of quantity data to the base year for expenditures and implicit quantities and drop the oldest month of quantity data for the augmented base year. The approximate GEKS indexes for the new augmented years are calculated in the usual way and the ratio of the index value for the last month in the new augmented year to the index value for the previous month in the new augmented year is used as an *update factor* for the value of the index for the last month in the previous augmented year. The resulting indexes are called *Approximate Rolling Year GEKS indexes*.

Since this method has not been suggested before, there have been no numerical experiments to see how it performs in practice. A priori, this method appears to be an improvement over traditional annual basket methods but it will have to be tested before it can be used by national statistical agencies in their consumer price indexes.

Some of the methods described in this section can be adapted to the elementary index context as will be seen in the following section.

[136] Thus the above GEKS procedure is an improvement over the suggestion of Feenstra and Shapiro (2003) who chose only a single base month.

8. The Rolling Year Time Product Dummy Method for Constructing Elementary Indexes

8.1 The Time Product Dummy (TPD) Method for Constructing Elementary Indexes

Recall the simple stochastic approach to elementary indexes that was explained in section 4.6 above. In that approach, a statistical model was proposed that used only the data of two consecutive periods. A problem with this approach is that if there are strongly seasonal commodities in the elementary aggregate, they can have no influence on the elementary aggregate unless the commodity is present in both periods under consideration. Thus it seems more appropriate to extend the model of section 4.6 to cover the data for an augmented year so that strongly seasonal commodities will have an influence on the elementary indexes. The basic hypothesis is that prices within the elementary stratum vary proportionally over time except for random errors. Thus if a sample of M items in the elementary stratum are priced over the thirteen months in the current augmented year, let S(c) denote the set of items that are actually priced during month c of the augmented year and denote the price of item m in month c of the augmented year by $p_m{}^c$ for c = 1,2,...,13 $m\in S(c)$ and c = 1,2,...,13. The counterparts to equations (74) and (75) in section 4.6 are the following equations:

(151) $p_m{}^1 \approx \beta_m$; $\quad m\in S(1)$;
(152) $p_m{}^c \approx \alpha^c\beta_m$; $\quad c = 2,3,...,13;\ m\in S(1)$.

The sequence of numbers 1, α^2, α^3,..., α^{13} are the desired factors of proportionality over the 13 months in the current augmented year and they represent the sequence of elementary indexes for the stratum under consideration for the augmented year. The parameters β_1, β_2,..., β_M represent quality adjustment factors that adjust for the differing quality of the outlets sampled and the items chosen. Adding multiplicative error terms to the right hand sides of (151) ad (152) and taking logarithms of both sides of the resulting equations leads to the following system of linear in parameters estimating equations:

(153) $\ln p_m{}^1 = b_m + \varepsilon_m{}^1$; $\quad m\in S(1)$;
(154) $\ln p_m{}^c = a^c + b_m + \varepsilon_m{}^1$; $\quad c = 2,3,...,13;\ m\in S(1)$

where $b_m \equiv \ln \beta_m$ for m = 1,...,M and $a^c \equiv \ln \alpha^c$ for c = 2,3,...,13.

Once the a^c parameters have been estimated, estimates of the α^c can be obtained by exponentiating the a^c. It can be seen that the model defined by (153) and (154) is formally identical to Summer's (1973) *Country Product Dummy* model for generating elementary indexes except that the time periods in the current augmented year replace the countries in Summer's method. Thus it seems appropriate to term the present time series model the *Time Product Dummy* model for generating elementary indexes.[137]

[137] Balk (1980c; 70) proposed a weighted version of this model in the time series context and Diewert (2004) proposed a weighted version of this model in the multilateral context.

Some of the advantages of the TPD method of constructing elementary aggregates over traditional method that rely on the Jevons, Carli or Dutot formulae as aggregation techniques are as follows:

- The TPD estimates do not single out a single month's prices as the base prices as is done using the current ONS methodology for the RPI that uses Carli indexes with a January base month; the TPD method treats all months in the current augmented year in a completely symmetric manner.
- The TPD method makes use of all of the price information collected for the current augmented year and so there is maximal (implicit) matching of prices within the augmented year.
- There is no need for special methods (like carry forward missing prices) to deal with missing prices or strongly seasonal commodities: the TPD method automatically deals with these problems.
- The TPD method generates standard errors for the resulting elementary indexes.
- The TPD method generates elementary indexes that satisfy Walsh's multiperiod identity test; i.e., if the prices for two months in the augmented year are identical, then the resulting TPD index values will be identical for those two months. Thus TPD indexes will be free of chain drift.

Of course, a problem with the TPD indexes is that they will change as data for a new month becomes available. In the following subsection, we use the Rolling Year methodology to overcome this difficulty.

8.2 The Rolling Year Time Product Dummy (RYTPD) Method for Constructing Elementary Indexes

As usual, the Rolling Year GEKS methodology explained in section 7.2 can be adapted to deal with the problem identified at the end of the last subsection. Thus we add the price data for the most recent month to the current augmented year and drop the price data for the oldest month in order to obtain a new augmented year for prices. A new model defined by (154) and (155) for the new augmented year is estimated and new index values say α^{12*} and α^{13*} are generated for the new augmented year. The value of the index for the last month in the previous augmented year is multiplied by $\alpha^{13*}/\alpha^{12*}$ in order to obtain an updated index for the last month in the new augmented year. The resulting indexes are called *Rolling Year Time Product Dummy*(RYTPD) *elementary indexes*.

Since this method has not been implemented, there have been no numerical experiments to see how it performs in practice.[138] A priori, this method appears to be an improvement

[138] Although the method has not been precisely implemented, Balk (1980c) more or less implemented a weighted version of the model using his Dutch data set on greenhouse fruits and vegetables for 72 months. Balk computed his weighted TPD indexes, starting with the first 4 months of data, then adding one more month of data and ending up using all 72 months of data. He found that once 48 months of data were used,

over traditional elementary index methods but it will have to be tested before it can be used by national statistical agencies in their consumer price indexes.

adding an additional month of data to the previous regression model did not significantly change the earlier model price levels. Thus for his data set, a rolling window of length 48 months would probably work well.

9. The Problem of Fashion Goods

A *fashion good* is a good which comes on the market with a price premium due to its newness and then declines in price as its "newness" wears off. Examples of fashion commodities are certain items of women's clothing, automobiles, electronic games and movies.

One of the best papers which deals with fashion goods in a systematic manner is by Greenlees and McClelland (2010). They obtained scanner data on apparel sales of "women's tops" in the US for a number of years and they tried a wide variety of techniques to deal with the problems associated with fashion goods. The basic problem is that at the beginning of the fashion season, a fashion good comes into the marketplace at a high price and then as time passes, the price of the same item declines rapidly. Thus if any kind of matched model price index is used, the resulting index will show a tremendous downward movement throughout the year. When a new fashion item that is somewhat comparable is introduced at the beginning of the following season is linked in with the last price of last year's comparable fashion item, and then the index will rapidly decline to a very low level. However, if one uses annual unit value prices for the fashion items, there is very little change in these prices over time.

How can a fashion good be detected? In the case of a clothing item, a useful test would be a persistent decline in the price of the item after its introduction. Greenlees and McClelland (2010) document the decline in price of a women's fashion item (misses' tops) for a major department store chain in the U.S. from the start of each March over the years 2004-2007. Only 2 percent of transactions in this item take place at the introductory price at the start of each season. The average selling price is about one half of the starting price. Non fashion items tend to sell at prices that may fluctuate but do not persistently decline.

There are two problems that are associated with the use of average prices to reflect the movements in the prices of fashion items:

- Average prices fail to account for any changes in average item characteristics or "quality" over time.
- The effects of the product cycle for these fashion goods leads to tremendous fluctuations in the month to month indexes. These fluctuations are not "real" because the same physical item in the middle of the year is not the same (in the eyes of purchasers) as the item when it is "newer".

The answer to the above problems is to treat each fashion good in each month of its life as a *separate good*. These separate goods cannot be directly compared across the months within the fashion cycle but a brand new car of a certain model and type can be compared with the comparable brand new car that is introduced at the start of the next fashion cycle. The effect of this treatment is to make each vintage of a fashion good a *strongly seasonal commodity* that can be compared across fashion seasons (for models of the same vintage) but cannot be directly compared within the fashion season.[139] Basically, comparing prices of a particular fashion item as it "ages" in the market place is not comparing like with like: an older vintage of a fashion good gives less utility to purchasers as compared to a new vintage. Index number theory rests on the assumption that we are comparing the same good (that gives purchasers the same utility) at two or more time periods and fashion goods do not satisfy this criterion: different vintages of the same physical fashion good give purchasers differing amounts of utility.

[139] Of course, there is a practical problem facing price collectors in following this advice: it may be difficult to determine if a particular fashion item just introduced in the current month is comparable to last season's just introduced fashion item.

10. Recommendations

Our recommendations for changing procedures will be directed at the Retail Prices Index (RPI) since the ONS has control over this index whereas the CPI or HICP is controlled by European legislation.[140]

10.1 Short Run Recommendations

The following three recommendations could be implemented in the near future.

Recommendation 1: The scope of the RPI should be changed to cover the expenditures of all households in the UK on consumer goods and services.

The present restrictions on the scope of the RPI seem to be a historical artefact and should be dropped.

Recommendation 2: The RPI should drop its use of the Carli index as an elementary index and replace it by either the Jevons index or the Carruthers Sellwood Ward and Dalen elementary index.

The upward bias in the Carli index was explained in sections 4.4 and 4.5 and illustrated in sections 4.8 and the Appendix. Arguments that attempt to justify the use of the Carli index on the basis of the economic approach to elementary indexes that are presented on pages 364-369 of the *Consumer Price Index Manual* cannot be used to justify the use of the Carli index. What the arguments in the *Manual* show is that *if* one has knowledge of the expenditure shares associated with each price, then a sampling scheme that uses this weighting information can recover various indexes such as the Laspeyres and Paasche indexes. However, at the elementary level, by definition, the price statistician does *not* have the required information on weights.[141] The use of Dutot elementary indexes is satisfactory if the elementary stratum is very narrowly defined so that all items in the stratum are very similar. But if this narrowness assumption is not satisfied, the use of Dutot indexes should be avoided; see section 4.8 for an example where the use of the Dutot index is not appropriate.

Recommendation 3: Fashion goods should not be used as priced items in the current RPI.

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[140] The HICP uses much the same methodology as the RPI and so many of the recommendations made in this section apply also to the HICP. One major difference between the RPI and HICP is that the HICP does not use the Carli index at the elementary level whereas the RPI uses the Carli in about 50% of its strata. This methodological difference probably explains most of the recent differences in the two indexes. It should be noted that the HICP was initially designed as a general inflation index (with no imputations) as opposed to a measure of domestic household consumer price inflation. For a discussion of the methodological differences between the HICP and a Consumer Price Index based on a cost of living framework, see Diewert (2002).

[141] In retrospect, it was probably a mistake to include this material on the economic approach to elementary indexes in the *Consumer Price Index Manual*.

If the overall methodology used in the RPI is changed to use monthly baskets and rolling year techniques, then fashion goods can be introduced into the list of sampled items but these items need to be treated as strongly seasonal products; see section 9 above.

Recommendation 4: The use of Young indexes (weighted Carli indexes) at lower levels of aggregation should be eliminated due to the inherent upward bias in these indexes.[142]

These Young indexes could be replaced by their weighted Jevons counterparts, which are not subject to the bias problem. Alternatively, the weighting structure used in the RPI could be simplified. At present, there are multiple layers of aggregation in the RPI and it is difficult to determine what the effects of these layers are (and it is difficult to explain to the public exactly how the aggregation is accomplished). Ideally, there would be only two layers of aggregation: a lower level of aggregation where there are no weights available and so an elementary index number formula would be used to aggregate the items within each stratum at this lower level and a higher level of aggregation, where weights are available.

10.2 Longer Run Recommendations

The following recommendations are longer run recommendations. Basically, these recommendations involve using new techniques to produce better indexes. The biggest problem with the current RPI and HICP is that they use monthly prices but annual weights. As was explained in section 5.4 above, this is not methodologically sound. But before these suggestions on producing more methodologically sound indexes are implemented, they need to be tested by the ONS; i.e., they need to be produced on an experimental basis and carefully evaluated before they are published.

Recommendation 5: The ONS should collect monthly expenditure information on household expenditures so that monthly expenditure shares for the target population can be calculated.

This recommendation would necessitate more resources for improving the existing consumer expenditure surveys. This recommendation is a key one. Existing approaches to the construction of consumer price indexes all assume that information on prices and the corresponding quantities consumed is available at the frequency of the index so if the index is produced on a monthly basis, information on both prices and quantities is required on a monthly basis.[143]

The following four recommendations assume that monthly expenditure and price information is available.

[142] See the ONS (2012; 41) for an explanation of how Young indexes are used at lower levels of aggregation in the RPI.
[143] See section 5.4 above for criticisms of approaches to index number theory that use monthly prices but annual quantities.

Recommendation 6: The Approximate Rolling Year GEKS indexes described in section 7.3 should be produced by the ONS on an experimental basis.

If these indexes perform well, then this index should become the headline measure of inflation and replace the current RPI.

Recommendation 7: The Rolling Year GEKS indexes described in section 7.2 should be produced on a delayed basis (as current month expenditure information becomes available) as an analytic series.

The approximate Rolling Year GEKS indexes will have a small amount of substitution bias in them and so in order to form estimates of this bias, it will be necessary to produce "true" Rolling Year GEKS indexes on a delayed basis in order to evaluate this bias.

Recommendation 8: The ONS should explore the possibility of obtaining detailed price and quantity data from the major supermarket chains in the UK.

Chains in the Netherlands and in Australia have provided this information to their national statistical agencies on a voluntary basis and it seems likely that this would be possible in the UK as well. With detailed price and quantity information, accurate elementary indexes for many strata could be calculated as the Dutch experience demonstrates; see section 7.2 above.

Recommendation 9: The ONS should compute Rolling Year Time Product Dummy elementary indexes on an experimental basis.

If the resulting experimental indexes look reasonable, then the ONS should replace its current methods for constructing elementary indexes by RYTPD indexes. The methodological advantages of these indexes are listed in section 8.2 above.

The final two recommendations are probably not as important as the previous recommendations. Basically, these two final recommendations boil down to an index number method for seasonally adjusting a consumer price index but other methods of seasonal adjustment may work just as well.

Recommendation 10: The ONS should produce the Approximate Rolling Year Annual indexes described in section 6.4 as analytic indexes.

Recommendation 11: As information on current period monthly expenditures becomes available, the ONS should produce the superlative Rolling Year Annual indexes described in section 6.4 on a delayed basis as analytic indexes.

These Rolling Year Annual indexes should be very useful to the Bank of England as seasonally adjusted measures of consumer price inflation and may be suitable as target indexes. The differences between the Approximate and True Annual Rolling Year indexes described in sections 6.4 should provide some information on the amount of higher level substitution bias that is inherent in the Approximate Rolling Year Annual indexes.

Many of the above recommendations should be considered as recommendations for further research by other national and international statistical agencies. In particular, Recommendations 6-11 can also be applied to the HICP.

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